A Computable Theory of Dynamic Congestion Pricing

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Overview

The advent of new commitments by municipal, state and federal governments to construct and operate roadways whose tolls may be set dynamically has brought into sharp focus the need for a computable theory of dynamic tolls. By a computable theory we mean a mathematical representation which is detailed enough to capture the key behavioral and technological considerations relevant to dynamic tolling and which is nonetheless numerically tractable enough to obtain accurate approximations of optimal dynamic toll trajectories.

It is clear from the policy debates that surround the issue of dynamic tolls that pure economic efficiency is not the sole or even the most prominent objective of any dynamic toll mechanism that will be implemented. Rather, equity considerations as well as preferential treatment for certain categories of commuters must be addressed by such a mechanism. Accordingly, we introduce in this paper the **d**ynamic **o**ptimal **t**oll **p**roblem with user **e**quilibrium **c**onstraints whose acronym will be DOTPEC. We provide a proof of existence for plausible regularity conditions. We also present two algorithms and test both on a numerical example.

Background

The dynamic user equilibrium optimal toll problem (DOTPEC) should not be confused with a dynamic extension of the traditional congestion pricing paradigm associated with static user equilibrium and usually credited to Beckmann et al (1956). Rather, the DOTPEC is closely related to the equilibrium network design problem which is now widely recognized to be a specific instance of a mathematical program with equilibrium constraints (MPEC).

The relevant background literature for the DOTPEC includes a paper by Friesz et al (2002) who discuss a version of the DOTPEC but for the day-to-day time scale rather than the dual (within-day as well as day-to-day) time scale formulation emphasized in this paper. Also pertinent is the paper by Friesz et al (1996) which discusses dynamic disequilibrium network design and the review by Liu (2004) which considers multi-period efficient tolls. In the final manuscript associated with this abstract, we will include our own review of the multi-period efficient toll problem as well as a review of the static equilibrium network design literature and the relevance of the ideas contained therein to the DOTPEC.

Comparison to Efficient Tolls

As noted above, the DOTPEC is not obviously equivalent to a dynamic generalization of the static efficient toll problem. In fact the exact nature of the differences and similarities of the two formulations is not known and has never been studied. As a consequence we will devote a section of

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the proposed paper to the formulation of the dynamic efficient toll problem and its numerical solution.

To study the dynamic efficient toll problem (DETP), it is necessary to employ some form of dynamic user equilibrium model. We elect the formulation due to Friesz et al (2001) and Friesz et al (2006) and its varieties analyzed by Ban et al (2006) and others. The dynamic DETP formulation will be constructed by direct analogy to the static efficient toll problem formulation of Hearn et al (2002).This approach to the formulation of the DETP leads directly to an the efficient toll pricing rule, provided appropriate necessary conditions that recognize time shifts are employed. The necessary conditions are those derived by Friesz et al (2004) for optimal control problems with state-dependent time shifts.

In particular the relevant dynamic user equilibrium problem is to find $h^* \in \Omega$ such that

$$\int_{t_0}^{t_f} \Psi\left(x\left(h^*\right), h^*, t\right)\left(h - h^*\right) dt \geq 0 \qquad (1)$$

for all $h \in \Omega$, where h and h^* are vectors of departure rates, and Ψ is a vector of effective path delay operators; additionally Ω is the set of state equations, flow propagation constraints and non-negativity constraints. The state variables are arc volumes, and the flow propagation constraints involve state-dependent time shifts. Note that (1) is equivalent to

$$\int_{t_0}^{t_f} \Psi\left(x\left(h^*\right), h^*, t\right) h dt \geq \int_{t_0}^{t_f} \Psi\left(x\left(h^*\right), h^*, t\right) h^* dt$$

which is tantamount to requiring that the vector of equilibrium departure rates solve the functional mathematical program

$$\min \int_{t_0}^{t_f} \Psi\left(x\left(h^*\right), h^*, t\right) h dt \quad \text{s.t. } h \in \Omega \qquad (2)$$

Because of the structure of Ω, problem (2) is an optimal control problem. Friesz et al (2001) showed that the necessary conditions for this problem is a user equilibrium. See Friesz and Mookherjee (2006) for a summary of the analysis given in Friesz et al (2001).

Furthermore, the system optimal dynamic traffic assignment problem is also an optimal control problem; in fact the system optimal problem takes the form

$$\min \int_{t_0}^{t_f} \Psi\left(x(h), h, t\right) h dt \quad \text{s.t. } h \in \Omega \qquad (3)$$

which is subtlely but importantly different than (3). When we introduce arc-specific congestion tolls, the vector of which we denote by y, it is convenient to write the system optimal problem as

$$\min \int_{t_0}^{t_f} \Psi\left(x(h), h, t; y\right) h dt \quad \text{s.t. } h \in \Omega \qquad (4)$$

where $\Psi(x(h), h, t; y)$ is the vector of effective path delay operators parameterized in terms of the arc-specific congestion tolls. Again (4) is an optimal control problem with state-dependent time shifts. Comparison of the DUE necessary conditions obtained from (2) with the system optimal congestion pricing necessary conditions obtained from (4) provides an efficient toll decision rule of the form

$$y = G(h) \qquad (5)$$

In the full paper we will present the detailed analysis leading to (5) as well as the precise functional form of (5). The decision rule (5) will be compared to solutions of the DOTPEC.

Formulation of the DOTPEC

The main focus of this paper is the formulation and solution of the DOTPEC. To this end, again using the DUE formulation reported in Friesz et al (2001) and Friesz and Mookherjee (2006), we will form a Stackelberg game that envisions a central authority minimizing social costs through its control of link tolls subject to DUE constraints with additional side constraints for equity and other policy considerations. Also, as we will allow multiple target arrival times of the users, the within-day scale model can be easily extended to include day-to-day time scale properly. Of course there are several ways such a model may be formulated. We shall present two formulations of the DOTPEC; these will be based on our work on differential variational inequalities and equilibrium network design.

For the first formulation, we recall that Friesz et al (2004) show that variational inequalities in infinite dimensional vector spaces with differential equations among their constraints have necessary conditions that take the form of two-point boundary value problems. This allows us to replace the infinite dimensional variational inequality representation of a DUE with a system of differential equations leading to a well defined optimal control problem that is equivalent to the DOTPEC.

For the second formulation, we recall that Tan et al (1979) show that a finite system of inequalities is equivalent to a static user equilibrium allowing -- as explained in Friesz and Shah (2001) -- the equilibrium network design problem to be expressed as a single level mathematical program. An extension of this result to the dynamic setting will allow us to express the DOTPEC as a non-hierarchical optimal control problem.

In general, the set of admissible controls of an MPEC, of which the DOTPEC is a special case, is not convex so that an existence theorem is difficult to construct. In that this paper is computationally focused, we do not intend to study DOTPEC existence.

Algorithms for the DOTPEC

We will also explore two principal methods for solving optimal control representations of the DOTPEC:

• descent in Hilbert space without time discretization, and

• a finite dimensional approximation solved as a nonlinear program.

In both algorithms we will employ an implicit fixed point scheme like that in Friesz and Mookherjee (2006) for dealing with time lags that arise in the flow propagation constraints. Our current plan is study a five arc, four node, two origin-destination pair example problem in detail. We are not presently planning to study the 76 arc Sioux Falls network. Examples based on that network and even larger problems will be studied in subsequent manuscripts.

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