

Optimal Contract Problems in Online Advertising with Risk Considerations

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Abstract

In this paper, we study optimal contract problems for online display advertisements with pay-per-view pricing scheme. We first provide and analyze a single contract model, which is shown to be equivalent to the newsvendor problem. We then consider a stochastic optimization problem with two different contracts and show that no mixed contract is optimal for risk-neutral publisher. However, we show that a mixed contract strategy may be optimal when we consider the risk attitude of the publisher. Numerical experiments illustrate the change of optimal strategy for different risk levels.

Keywords: online advertising, optimal contract, risk aversion.

1 Introduction

Online advertising has already become a dominant ad-medium and is continuously gaining market share. In the United States, with 2008 revenue of \$23.4 billion, online advertising is now the third largest ad-supported medium, marking increasing significance to marketers and consumers (Interactive Advertising Bureau, 2009). Online advertising in the US is expected to grow eight times faster than the overall market and would be the second largest behind direct marketing with a total yearly revenue of \$51.1 billion by 2012 (Information Week, 2008). According to Interactive Advertising Bureau (2009), search and display are the major formats of online advertising having 77% market share. Search advertisement appears as a search result in a web page loaded in response to a user keyword(s) search request. Display advertisement, on the other hand, is displayed in web-pages which are requested for different purposes. With about 45% of the total market share, search is already the leading format and is expected to have a larger share. At the same time, an increasing number of advertising efforts would intend to focus in achieving memory rather than instantaneous response. So, display will always be a major mode of advertising.

There are several pricing methods for charging online advertising services. Among those, pay-per-view (PPV) and pay-per-click (PPC) are dominant in the market. In particular, PPV is popular in display advertising and PPC is popular in search-based keyword advertising. However, there are two major sources of uncertainty in online advertising, one is the number of page-views and the other is the number of clicks. In this paper, we consider the case that the

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publisher has decided to host only display advertisements that use pay-per-view pricing scheme or the nature of the problem suggests so. We will examine how to make contracts optimally while the number of page-views is uncertain. Also, the publisher needs to determine the contract mix when she has multiple display advertisements to choose from. We study the optimal contract problem for single as well as two PPV advertisements where the publisher is *risk-neutral*. We also perform similar study of optimal contract decision problems when the publisher is willing to take a specified risk or to be *risk-averse*.

Regarding online advertising, there are not many research reported in the literature of operations research. Mangani (2004) addressed optimal decision making between cost-per-impression (CPM) and cost-per-click (CPC) advertisements when the publishers are price-takers in the display advertisement market and later Fjell (2009) revisited the same problem. Both Mangani (2004) and Fjell (2009) used the concept of elasticity to reach the conclusions. Kwon (2009) also studied a capacity allocation problem between CPM and CPC advertisements with stochastic page-view and click-through rate (CTR) and provided a stochastic optimization formulation. Kumar and Sethi (2009) considered a dynamic pricing problem considering subscription and advertising. Roles and Fridgeirdottir (2009) studied a dynamic optimal customer selection and display scheduling problem considering only CPM contracts. Fjell (2010) first considered the problem for a publisher who has the market power to set the price. Fridgeirdottir and Naja-Asadolahi (2009a,b) also studied revenue management problems with cost-per-impression (CPM) contracts in consideration. However, to the knowledge of the authors, no study on optimal contract problems has been reported.

Stochastic inventory problems with risk consideration have been studied for a while. A good number of these studies consider Value-at-Risk (VaR) as the measure of risk. Özler et al. (2009) studied single, two and multi-product newsvendor problem incorporating VaR risk measure. The exact formulation of the problem for single and two products case has been studied. They also found the results for two products for the case of correlated demand. An approximation method has been proposed to obtain the results for a multi-products case. Luciano et al. (2003) also studied multi-period static inventory models with VaR being the risk measure. Zhou et al. (2008) proposed an optimal-order model to consider multi-product inventory problem with Conditional Value-at-Risk (CVaR) constraints. The model is simulated for the case of a newsvendor problem and they find the solution bound is fully consistent with the decision maker's intuition on return-risk decision making. The return-CVaR model is found to be more flexible than the classical model. YANG et al. (2008) considered the risk of a newsvendor with limited capacity. Both the downside risk measure and CVaR risk measure has been used here to optimize the model. Tapiero (2005) also addressed stochastic inventory control with VaR approach. He showed that VaR approach is justified by a disappointment criterion and provides its applications to inventory management.

In this work, *display* of an advertisement refers to loading a web-page containing the advertisement in response to a online user request to load that page. When a page containing an advertisement is displayed, we count that as a single *impression*. The number of *page-views* is a random variable specifying how many times a publisher's web-page containing an advertisement is requested by visitors. The term *capacity* is used synonymously with the number of page-views, in the perspective of web publishers for online advertising services. After the period of consideration is over, the total number of display of an advertisement is termed as *realized display* of that advertisement. For displaying only one advertisement, realized display and realized number of page-views are same. If the publisher displays more than one advertisement, she needs to

decide on allocation and *display rule*. The publisher may display an advertisement first until the contracted number of displays, and then start displaying another advertisement. We call this display rule a sequential display rule. On the other hand, the publisher may display advertisements proportionally. That is, the publisher displays advertisements in a mixed sequence of various advertisements. We call this display rule a proportional display rule. In this paper, we will also consider proportional display rule.

The paper is organized as follows. Section 2 provides the optimal contract problem for a single contract, and provides the equivalence of the optimal contract problem to the well-known newsvendor problem. Then, Section 3 considers an optimal contract problem with two display advertisements contracts and shows that no mixed contract is optimal. Section 4 formulates the problems for single and two PPV advertisements cases considering the risk compared with risk neutral strategy which is followed by numerical examples in Section 5. The numerical results show that when publisher is willing to take a high risk in generating more revenue, a mixed contract strategy is optimal in certain cases. Finally, Section 6 concludes the paper.

2 Optimal Contract Design Problem: Single PPV Advertisement

In this section, we will develop a model for the optimal contract problem in online advertising. In particular, we will consider a contract problem for displaying online advertisements with pay-per-view (PPV) pricing scheme. Payment for PPV is dependent on cost-per-impression (CPM), hence the revenue of a publisher depends on how many impressions she can be able to display.

When we consider a publisher's optimal contract problem, two types of advertisements have PPV pricing scheme: one is advertisements by direct contract with the publisher, and the other is so-called network advertisements (Roles and Fridgeirsdottir, 2009). The former type, henceforth termed as PPV advertisements, involves a contract that specifies how many times the advertisements will be displayed within a certain time-frame. Network advertisements are sourced through third-parties or network agencies who makes the contact with the advertisers. These network advertisements are available for a flat rate lower price. The publisher generally tends to display network advertisements to fulfill the unused capacity after displaying the PPV advertisements.

When the publisher makes a direct contract with an advertiser to display a PPV advertisement, she promises a certain number of displays. For any reason, when the publisher cannot display the contracted number, she has two options. She may consider refund to the advertiser for un-displayed numbers, or she may carry them over to the next time period. For the former case, the unit refund would be greater than or equal to the marginal revenue collected at the time of contract. We call this *direct* penalty. For the latter case, which we call *indirect* penalty, the publisher may lose potential contracts because of the carried-over. In addition, there could be loss of goodwill in both cases. Therefore, there is a penalty when the contracted number of displays is not fulfilled. However, for network advertisements, there is no such promise on the number of displays.

As the direct or indirect penalty is an important component of the operation of online advertising systems, the number of page-views for the future time period is critical to the publisher. A page-view is defined as a request to load a single page on a visitor's screen. The total number of page-views is definitely subject to uncertainty, i.e., the publisher can only have a forecast on this value. We use a random variable X to denote the uncertain number of page-views

during the time period of interest.

In this section, we assume the publisher has only one web-page to display and assume only one advertisement can be displayed at a time. Also, the publisher will either make a single contract, or make contracts with many advertisers who are homogeneous (same CPM and same penalty). We provide a formulation to consider the single contract problem or the aggregate contract problem. Our objective is to determine the optimal number of displays that should be promised to the advertiser(s). Let v denote the aggregate number of displays to promise (decision variable). We use p and h for the homogeneous CPM and penalty, respectively. Although CPM is the cost for a thousand impressions, we use p for one impression, for simplicity. Then, with the page-view X , the revenue of the publisher is:

$$R(v) = pv - h \max\{(v - X), 0\} + q \max\{(X - v), 0\} \quad (1)$$

where q is the unit revenue from network advertisements. If there are extra page-views after fulfilling all contracts, the publisher uses those extra capacities for network advertisement. When we have the distribution information of X , the optimization problem

$$\max_v \mathbb{E}[R(v)] \quad (2)$$

is a simple stochastic optimization problem. Its solution is easily obtained as:

$$v^* = F^{-1} \left(\frac{p - q}{h - q} \right) = F^{-1} \left(\frac{\bar{p}}{\bar{p} + \bar{h}} \right) \quad (3)$$

where $F(\cdot)$ is the cumulative distribution function of X , $\bar{p} = p - q$ and $\bar{h} = h - p$. We make a reasonable assumption $h > p > q$ so that \bar{p} and \bar{h} are positive.

We can show the relationships between the optimal contract problem (2) and the well-known newsvendor problem (Petruzzi and Dada, 1999). The main difference between the two problems is that demand information is uncertain in the newsvendor problem, while capacity (page-views) is uncertain in the optimal contract problem (2).

We consider a newsvendor who has to make decision everyday about the amount of paper Q he would be ordering before start delivering the paper and observe the realized demand D . The purchase cost of paper is c per paper and he sells them at a rate of π per paper. Though the actual demand D is an unknown *a priori*, the newsvendor has a fairly correct idea of the distribution pattern of D . Let $G(\cdot)$ and $g(\cdot)$ respectively denote the cumulative distribution function (cdf) and probability density function (pdf) of D . At the end of the sales, if the actual demand is less than the order quantity, the excess papers are returned and he receives the salvage price s per paper. We assume $\pi > c > s$, while s can be negative. If the actual demand is more than the order quantity, the newsvendor loses the potential profit he could be making by selling the additional number of products. Also, he might lose potential customers in the future. To account for this issue, cost of lost opportunity of selling is assumed to be l per unit. The newsvendor problem can be formulated as following:

$$\max_Q \mathbb{E}[Z(Q)] = \mathbb{E}[\pi \min\{Q, D\} - cQ + s \max\{(Q - D), 0\} - l \max\{(D - Q), 0\}] \quad (4)$$

and its solution is

$$Q^* = G^{-1} \left(\frac{\pi + l - c}{\pi + l - s} \right) = G^{-1} \left(\frac{\bar{\pi}}{\bar{\pi} + \bar{c}} \right) \quad (5)$$

where $\bar{\pi} = \pi + l - c$ and $\bar{c} = c - s$.

The optimal contract problem (2) has a very similar structure to the newsvendor problem (4), and in fact they are equivalent. A proof of the following proposition is provided in Appendix.

Proposition 1. The optimal contract problem (2) is a newsvendor problem with the sales price π , the purchase cost $\pi - p$, the salvage value $\pi - h$, the lost opportunity cost $-q$ and the demand X , for any arbitrary π such that $\pi > h > p > q$.

We further note that the underage cost and the overage cost are, respectively:

$$\begin{aligned}\bar{\pi} &= \pi + l - c = p - q = \bar{p} \\ \bar{c} &= c - s = h - p = \bar{h}\end{aligned}$$

For a risk-neutral publisher, the optimal contract size sets the upper limit so that she can decide the number of display of advertisement she should commit. If the publisher announces to accept advertisement at CPM (p), many advertisers may be willing to make contract which is synonymous to making contracts with homogeneous advertisers. Even if the aggregate demand of the advertisers is higher than her optimal contract size, the publisher should reject contract offers after her aggregated size of contracts reaches optimal contract size.

3 Optimal Contract Design Problem: Two PPV Advertisements

The single contract problem (2) with the homogeneity assumption is simple and equivalent to the well-known newsvendor problem. However, when we take account more than one heterogeneous PPV contracts into the optimal contract problem, the model becomes quickly complicated. Again, we consider two PPV contracts that are involved with the same page and assume only one advertisement can be displayed at a time. Therefore, it is important to decide the display order of the two PPV advertisements. If the publisher has enough page-views so that no PPV contract invokes penalty, the display order is not important. On the other hand, if she does not have enough number of page-views to display the contracted numbers, the display order becomes important, because the amount of penalty she pays depends on the display order. If she decides to display the advertisement with highest CPM (p_i) first, the other advertiser may find this unfair which may lead to loss of future business. Also, if the advertisement with higher penalty (h_i) is displayed later, the publisher might end up paying too much as penalty if the page-view is too low. To avoid this, if the publisher chooses to display advertisement with higher penalty first, she might lose potential earning if page-view fails to meet the contract obligation marginally. At least for the sake of fairness, we consider proportional display rule (d_1, d_2), so that the *realized* number of displays of advertisement i becomes:

$$\min \left\{ v_i, \frac{d_i}{d_1 + d_2} X \right\}$$

where v_i is the optimal number of displays for PPV advertisement i and X is the (stochastic) number of page-views. Since the above relation uses ratio, the unit of d_i is not important. The display rule (d_1, d_2) would be implemented in sequence. For example, if $d_1 = 3$ and $d_2 = 5$, any consecutive 8 page-views will constitute three displays of advertisement 1 and five of

advertisement 2. With display rule (d_1, d_2) , the revenue becomes:

$$R(v_1, v_2) = p_1 v_1 + p_2 v_2 - h_1 \max \left[v_1 - \max \left\{ \frac{d_1}{d_1 + d_2} X, X - v_2 \right\}, 0 \right] \\ - h_2 \max \left[v_2 - \max \left\{ \frac{d_2}{d_1 + d_2} X, X - v_1 \right\}, 0 \right] + q \max[X - (v_1 + v_2), 0] \quad (6)$$

where we may assume $d_i = p_i$, $d_i = p_i + h_i$, $d_i = h_i$ or any values. However, in this paper, we only consider $d_i = v_i$, for a couple of reasons. First, it is fair to advertisers. With $d_i = v_i$, if the realized number of page-views is 10% less than what is expected, each advertisement is displayed 10% less times than the contracted number of display regardless the price. Second, the formulation (6) becomes simpler so that it is more convenient to analyze. In particular, when $d_i = v_i$, objective function (6) reduces to:

$$R(v_1, v_2) = p_1 v_1 + p_2 v_2 - h_1 \max \left[v_1 - \frac{v_1}{v_1 + v_2} X, 0 \right] \\ - h_2 \max \left[v_2 - \frac{v_2}{v_1 + v_2} X, 0 \right] + q \max[X - (v_1 + v_2), 0] \quad (7)$$

The expected value of (7) is:

$$\mathbb{E}[R(v_1, v_2)] = p_1 v_1 + p_2 v_2 + q\mu - \{(h_1 v_1 + h_2 v_2) - q(v_1 + v_2)\}F(v_1 + v_2) \\ - q(v_1 + v_2) + \left(\frac{h_1 v_1}{v_1 + v_2} + \frac{h_2 v_2}{v_1 + v_2} - q \right) \int_0^{v_1 + v_2} x f(x) dx \quad (8)$$

Finally, the optimal contract problem for the two PPV advertisements becomes:

$$\max_{v_1, v_2 \geq 0} J = \mathbb{E}[R(v_1, v_2)] \quad (9)$$

Note that, unlike in the aggregate optimal contract problem, for multiple PPV cases, there is no equivalence to the newsvendor problem. Without loss of generality, we assume $p_1 > p_2$. In this paper, we only consider $h_1 > h_2$. If $p_1 > p_2$ and $h_1 < h_2$, the ad 1 is always better than ad 2; therefore, we need not consider such case. We let λ_1 and λ_2 denote the dual variables for the non-negativity constraints.

Considering the Karush-Kuhn-Tucker (KKT) conditions, we will investigate the following cases separately:

	$v_1 > 0$	$v_1 = 0$
$v_2 > 0$	Case 1	Case 3
$v_2 = 0$	Case 2	Case 4

When we find solutions of the KKT conditions in each case, we will have *candidates* for the optimum. Let us assume h_1 and h_2 are not very large, so that Case 4 is never optimal. Also note that, without a convexity of J , KKT conditions only give candidates for local optimum. Before we begin, we first obtain the following necessary condition:

Lemma 1. The following condition is necessary for an optimal solution to (9):

$$F(v_1 + v_2) = \frac{\bar{p}_1 v_1 + \bar{p}_2 v_2}{(\bar{h}_1 + \bar{p}_1) v_1 + (\bar{h}_2 + \bar{p}_2) v_2} \quad (10)$$

where $\bar{p}_i = p_i - q$ and $\bar{h}_i = h_i - p_i$.

Proofs to Lemma 1 and other subsequent lemmas and theorems are provided in Appendix. If we compare (10) with (3), we see that the total number of contract ($v_1 + v_2$) is relevant to a weighted ratio of \bar{p}_i to $\bar{h}_i + \bar{p}_i$. Let us introduce additional notations and definitions:

$$\begin{aligned} M(y) &= \frac{1}{y} \int_0^y x f(x) dx, & g(y) &= F(y) - M(y), & \Delta &= (\bar{h}_1 + \bar{p}_1) - (\bar{h}_2 + \bar{p}_2), \\ CR_1 &= \frac{\bar{p}_1}{\bar{h}_1 + \bar{p}_1}, & CR_2 &= \frac{\bar{p}_2}{\bar{h}_2 + \bar{p}_2}, & CR_{12} &= \frac{\bar{p}_1 - \bar{p}_2}{(\bar{h}_1 + \bar{p}_1) - (\bar{h}_2 + \bar{p}_2)} \end{aligned}$$

Note that $\Delta = (h_1 - p_1 + p_1 - q) - (h_2 - p_2 + p_2 - q) = h_1 - h_2 > 0$ by the assumption. Hence we have the following relation:

$$\begin{aligned} CR_{12} < CR_1 < CR_2 & \quad \text{if } \bar{h}_1/\bar{h}_2 > \bar{p}_1/\bar{p}_2 \\ CR_2 < CR_1 < CR_{12} & \quad \text{if } \bar{h}_1/\bar{h}_2 < \bar{p}_1/\bar{p}_2 \end{aligned} \tag{11}$$

For each case, we obtain the following result:

Lemma 2 (Case 1: $v_1 > 0, v_2 > 0$). Let k be a solution of the equation $g(k) = CR_{12}$. If such k exists, it is unique. Suppose the following condition holds:

$$\begin{aligned} CR_1 < F(k) < CR_2 & \quad \text{if } \bar{h}_1/\bar{h}_2 > \bar{p}_1/\bar{p}_2 \\ CR_2 < F(k) < CR_1 & \quad \text{if } \bar{h}_1/\bar{h}_2 < \bar{p}_1/\bar{p}_2 \end{aligned}$$

Then the following pair:

$$\begin{aligned} v_1^* &= \left(\frac{CR_1 - CR_{12}}{CR_2 - CR_1} \right) \left(\frac{CR_2 - F(k)}{F(k) - CR_{12}} \right) k \\ v_2^* &= \left(\frac{CR_2 - CR_{12}}{CR_2 - CR_1} \right) \left(\frac{F(k) - CR_1}{F(k) - CR_{12}} \right) k \end{aligned}$$

satisfies the KKT conditions. Therefore, it is a unique candidate in Case 1 for local optimum.

While Cases 2 and 3 reduce to a single PPV ad case, considering them in the KKT conditions leads to additional insights to the problem. We provide Lemmas 3 and 4 for Cases 2 and 3, respectively.

Lemma 3 (Case 2: $v_1 > 0, v_2 = 0$). If the following condition is satisfied:

$$M(\rho_1) \geq CR_1 - CR_{12} \tag{12}$$

where $\rho_1 = F^{-1}(CR_1)$, then $v_1^* = \rho_1$ and $v_2^* = 0$ is a unique KKT candidate for local optimum in Case 2.

Note that when $\bar{h}_1/\bar{h}_2 < \bar{p}_1/\bar{p}_2$, we automatically have $CR_1 - CR_{12} < 0$ by equation (11), and therefore, condition (12) is always satisfied. That is, when the advertisement 1 has higher relative revenue \bar{p}_1/\bar{p}_2 than penalty \bar{h}_1/\bar{h}_2 , it is always optimal that the publisher make a contract only with the advertisement 1.

Lemma 4 (Case 3: $v_1 = 0, v_2 > 0$). Let us have ρ_2 such that $F(\rho_2) = CR_2$. If we have

$$M(\rho_2) \leq CR_2 - CR_{12} \quad (13)$$

then $v_1^* = 0$ and $v_2^* = F^{-1}(CR_2)$ satisfy the KKT conditions; hence a unique candidate for local optimum in Case 3.

We note that when $\bar{h}_1/\bar{h}_2 < \bar{p}_1/\bar{p}_2$, we automatically have $CR_2 - CR_{12} < 0$ by equation (11), therefore, the condition (13) is never satisfied. This is consistent with the discussion of Lemma 3.

As Lemmas 2, 3 and 4 only give candidates for optimum, we need to compare the objective function values to identify the global solution. We obtain the expected revenue at the local solution of Case 1:

$$\begin{aligned} \mathbb{E}[R]_{\text{Case 1}} &= \bar{p}_1 v_1 + \bar{p}_2 v_2 + q\mu - [(\bar{h}_1 + \bar{p}_1)v_1 + (\bar{h}_2 + \bar{p}_2)v_2]CR_{12} \\ &= q\mu + \frac{\bar{p}_2 \bar{h}_1 - \bar{p}_1 \bar{h}_2}{\Delta} (v_1 + v_2) \\ &= q\mu + \frac{\bar{p}_2 \bar{h}_1 - \bar{p}_1 \bar{h}_2}{\Delta} k \end{aligned} \quad (14)$$

where $g(k) = CR_{12}$. For Cases 2 and 3, we obtain:

$$\begin{aligned} \mathbb{E}[R]_{\text{Case 2}} &= \bar{p}_1 v_1 + q\mu - (\bar{h}_1 + \bar{p}_1)v_1(F(v_1) - M(v_1)) \\ &= q\mu + (\bar{h}_1 + \bar{p}_1)v_1 M(v_1) \\ &= q\mu + (\bar{h}_1 + \bar{p}_1) \int_0^{F^{-1}(CR_1)} x f(x) dx \end{aligned} \quad (15)$$

and

$$\mathbb{E}[R]_{\text{Case 3}} = q\mu + (\bar{h}_2 + \bar{p}_2) \int_0^{F^{-1}(CR_2)} x f(x) dx \quad (16)$$

Note that (14) is only dependent on k . Using this fact, we provide the following proposition:

Proposition 2. The solution from Case 1, as in Lemma 2, is not a local optimum.

We make an important point: for any case, mixed contract ($v_1 > 0, v_2 > 0$) is not optimal. Therefore, either Case 2 or Case 3 (but not both) is optimal. A mixed contract is optimal only when two advertisements are identical, i.e., $p_1 = p_2$ and $h_1 = h_2$. That is, publisher does not need to consider differentiated contracts (Case 1).

The optimal solution can be determined by comparing (15) and (16). When $\bar{h}_1/\bar{h}_2 < \bar{p}_1/\bar{p}_2$, we know Case 3 is never optimal. Therefore, we only consider when $\bar{h}_1/\bar{h}_2 < \bar{p}_1/\bar{p}_2$. Comparing (15) and (16), we obtain that Case 2 is optimal if

$$\frac{\int_0^{F^{-1}(CR_1)} x f(x) dx}{\int_0^{F^{-1}(CR_2)} x f(x) dx} > \frac{CR_1 - CR_{12}}{CR_2 - CR_{12}} \quad (17)$$

and Case 3 is optimal otherwise. Table 1 summarizes these results for Case 2 and Case 3 based on the parameters and condition specified in equation (17). Note that only for parameter set

Parameter Set	Condition 1	Condition 2	Risk-Neutral Solution
i	$\frac{\bar{p}_1}{\bar{p}_2} \geq \frac{\bar{h}_1}{\bar{h}_2}$		$v_1^* = F^{-1} \left\{ \frac{\bar{p}_1}{\bar{p}_1 + \bar{h}_1} \right\}, v_2^* = 0$
ii	$\frac{\bar{p}_1}{\bar{p}_2} < \frac{\bar{h}_1}{\bar{h}_2}$	$\frac{\int_0^{F^{-1}(CR_1)} xf(x)dx}{\int_0^{F^{-1}(CR_2)} xf(x)dx} \geq \frac{CR_1 - CR_{12}}{CR_2 - CR_{12}}$	$v_1^* = F^{-1} \left\{ \frac{\bar{p}_1}{\bar{p}_1 + \bar{h}_1} \right\}, v_2^* = 0$
iii	$\frac{\bar{p}_1}{\bar{p}_2} < \frac{\bar{h}_1}{\bar{h}_2}$	$\frac{\int_0^{F^{-1}(CR_1)} xf(x)dx}{\int_0^{F^{-1}(CR_2)} xf(x)dx} < \frac{CR_1 - CR_{12}}{CR_2 - CR_{12}}$	$v_1^* = 0, v_2^* = F^{-1} \left\{ \frac{\bar{p}_2}{\bar{p}_2 + \bar{h}_2} \right\}$

Table 1: The optimal solution for different conditions

(iii), Case 2 ($v_1^* = 0, v_2^* > 0$) is the optimal solution. For the other two parameter sets, the optimal outcome is always Case 3 ($v_1^* > 0, v_2^* = 0$).

The results obtained in this section can be used to consider multiple PPV case. When the publisher have multiple contracts, she can make pairwise comparisons based on the decision rule specified in Table 1 and eliminate the inferior advertisement. Then she will pick another advertisement and make similar decision in the same way. In this way, the publisher will be able to select the desired advertisement and subsequently can decide on the optimal contract size.

In the following section, we will show a mixed strategy can be optimal when we consider a risk attitude of the publisher.

4 Optimal Strategy Considering Risk

The contract size specified in equation (3) is optimal when the publisher is risk-neutral. However, the publisher's risk attitude might be different, i.e., risk seeking or risk-averse. In this section, we study the publisher's risk for not choosing risk-neutral optimal contract size. We find the optimal contract number for a specified level of risk.

4.1 Single PPV advertisement

The publisher considers to adopt a strategy based on risk neutral contract size, i.e., $v^* = F^{-1} \left(\frac{p-q}{h-q} \right)$. Accordingly, she makes a contract to display the PPV advertisement v^* number of times. The revenue for this *Risk Neutral Strategy* will be:

$$R(v^*) = pv^* - h \max(v^* - X, 0) + q \max(X - v^*, 0) \quad (18)$$

The publisher would only want to choose a contract size other than v^* , if the strategy is expected to generate more revenue than the *Risk Neutral Strategy* with a probabilistic risk she is willing to accept. Let us consider the probability that a strategy v generates higher revenue than *Risk Neutral Strategy* with a probability of at least $(1 - \gamma)$. In particular, we consider the risk ϕ such that:

$$\phi = \Pr[R(v) \leq R(v^*)] \leq \gamma \quad (19)$$

Here, $R(v)$ and $R(v^*)$ is defined by equations (1) and (18) respectively. For a risk-averse publisher the value of γ is lower while γ is higher for a risk-seeking publisher. The values of ϕ can be

found by evaluating the following expression:

$$R(v) - R(v^*) = p(v - v^*) - h\{\max(v - X, 0) - \max(v^* - X, 0)\} + q\{\max(X - v, 0) - \max(X - v^*, 0)\} \quad (20)$$

Depending on the range of X , the value of right hand side of equation (20) is:

$$R(v) - R(v^*) = \begin{cases} \bar{h}(v^* - v) & \text{if } X \leq \min(v^*, v) \\ (\bar{h} + \bar{p})X - \bar{h}v - \bar{p}v^* & \text{if } v^* < X \leq v \\ -X(\bar{h} + \bar{p}) + \bar{p}v + \bar{h}v^* & \text{if } v < X \leq v^* \\ \bar{p}(v - v^*) & \text{if } \max(v^*, v) < X \end{cases}$$

where $\bar{p} = p - q$ and $\bar{h} = h - p$. Suppose, $v > v^*$, then we have

$$R(v) - R(v^*) = \begin{cases} \bar{h}(v^* - v) & \text{if } X \leq v^* \\ (\bar{h} + \bar{p})X - \bar{h}v - \bar{p}v^* & \text{if } v^* < X \leq v \\ \bar{p}(v - v^*) & \text{if } v < X \end{cases}$$

Therefore

$$R(v) - R(v^*) \leq 0 \quad \text{if } X \leq \frac{\bar{h}v + \bar{p}v^*}{\bar{h} + \bar{p}}$$

$$R(v) - R(v^*) > 0 \quad \text{if } X > \frac{\bar{h}v + \bar{p}v^*}{\bar{h} + \bar{p}}$$

Consequently, we obtain

$$\phi(v) = \Pr \left[X \leq \frac{\bar{h}v + \bar{p}v^*}{\bar{h} + \bar{p}} \right] = F \left(\frac{\bar{h}v + \bar{p}v^*}{\bar{h} + \bar{p}} \right) \quad \text{if } v > v^*$$

With similar analysis, for single PPV advertisement, the exact expression for equation (19) becomes:

$$\phi(v) = \begin{cases} F \left(\frac{\bar{h}v + \bar{p}v^*}{\bar{h} + \bar{p}} \right) & \text{if } v > v^* \\ 1 - F \left(\frac{\bar{h}v^* + \bar{p}v}{\bar{h} + \bar{p}} \right) & \text{if } v < v^* \\ 1 & \text{if } v = v^* \end{cases} \quad (21)$$

Note that the value of ϕ decreases as $v \rightarrow v^*$ because we can easily verify that:

$$\frac{\partial \phi}{\partial v} < 0 \quad \text{for all } v < v^*$$

$$\frac{\partial \phi}{\partial v} > 0 \quad \text{for all } v > v^*$$

Also, we find the cut-off probability:

$$\gamma_{\min} = \min \left\{ \lim_{v \rightarrow v^{*+}} \phi, \lim_{v \rightarrow v^{*-}} \phi \right\} = \min \{F(v^*), 1 - F(v^*)\} = \left\{ \frac{\bar{p}}{\bar{h} + \bar{p}}, \frac{\bar{h}}{\bar{h} + \bar{p}} \right\}$$

Accordingly, the publisher can never be able to earn more revenue than the *Risk Neutral Strategy* with a probability of more than $(1 - \gamma_{\min})$. As the parameters p, h, q or \bar{p}, \bar{h} are constant, the value of γ_{\min} remains constant irrespective of probability distribution of page-view X .

Consider the publisher wants to maximize revenue through making a contract v^* such that the revenue is higher than that generated through *Risk Neutral Strategy* with a probability of at least $(1 - \gamma)$ or $\phi \leq \gamma$. The problem can be written in terms of probability density and cumulative distribution function of v as:

$$\max_v \mathbb{E}[R(v)] = \bar{p}v + q\mu - (\bar{h} + v)vF(v) + (\bar{h} + \bar{p}) \int_0^v Xf(X)dx \quad (22)$$

subject to

$$\phi(v) \leq \gamma \quad (23)$$

where ϕ is defined by (21).

The problem (22) for single PPV advertisement is convex while the feasible space defined by the constraint (23) is piece-wise convex. In particular, the feasible space has three convex segments i.e., for $v \in [0, v^*)$, $v = v^*$ or $v > v^*$. The optimal solutions to problem (22) can be obtained by finding and comparing solution obtained by considering each convex segment of the feasible space.

4.2 Two PPV advertisements

In the case of two PPV advertisements, we find that the risk-neutral publisher should make contract to display only one of the advertisements following the guidelines provided in Table 1. In this section, we consider a *Risk Neutral Strategy* based on the above finding and find the optimal decision as the risk-attitude of the publisher changes. Accordingly, we propose a *Risk Neutral Strategy* for which the revenue will be:

$$R(v_1^*, v_2^*) = c_1 p_1 v_1^* + c_2 p_2 v_2^* - c_1 h_1 \max(v_1^* - X, 0) - c_2 h_2 \max(v_2^* - X, 0) + q \max(X - c_1 v_1^* - c_2 v_2^*, 0) \quad (24)$$

where $(c_1 = 0, c_2 = 1)$ while risk neutral solution, according to Table 1, is $(v_1^* = 0, v_2^* > 0)$ and $(c_1 = 1, c_2 = 0)$ otherwise.

Similar to the single PPV advertisement case, we consider the probability ϕ that an arbitrary strategy (v_1, v_2) generates no more revenue than the *Risk Neutral Strategy*:

$$\phi = \Pr[R(v_1, v_2) \leq R(v_1^*, v_2^*)]$$

where, $R(v_1^*, v_2^*)$ and $R(v_1, v_2)$ is defined by equations (24) and (6).

The values of ϕ can be obtained by considering $R(v_1, v_2) - R(v_1^*, v_2^*)$ with the range of X :

$$R(v_1, v_2) - R(v_1^*, v_2^*) = \begin{cases} -\left[\bar{h}_1(v_1 - c_1v_1^*) + \bar{h}_2(v_2 - c_2v_2^*)\right] \\ + \frac{h_1v_1 + h_2v_2 - (c_1h_1 + c_2h_2)(v_1 + v_2)}{v_1 + v_2} X & \text{if } X \leq \min[(v_1 + v_2), c_1v_1^* + c_2v_2^*] \\ -(\bar{h}_1v_1 + \bar{h}_2v_2 + c_1\bar{p}_1v_1^* + c_2\bar{p}_2v_2^*) \\ + \frac{(\bar{h}_1 + \bar{p}_1)v_1 + (\bar{h}_2 + \bar{p}_2)v_2}{v_1 + v_2} X & \text{if } c_1v_1^* + c_2v_2^* < X \leq (v_1 + v_2) \\ \bar{p}_1v_1 + \bar{p}_2v_2 + c_1\bar{h}_1v_1^* + c_2\bar{h}_2v_2^* \\ - (c_1h_1 + c_2h_2 - q)X & \text{if } (v_1 + v_2) < X \leq c_1v_1^* + c_2v_2^* \\ \bar{p}_1v_1 + \bar{p}_2v_2 - (c_1\bar{p}_1v_1^* + c_2\bar{p}_2v_2^*) & \text{if } \max[(v_1 + v_2), c_1v_1^* + c_2v_2^*] < X \end{cases}$$

Let us define

$$\begin{aligned} A &= \frac{(v_1 + v_2) [\bar{h}_1(v_1 - c_1v_1^*) + \bar{h}_2(v_2 - c_2v_2^*)]}{h_1v_1 + h_2v_2 - (c_1h_1 + c_2h_2)(v_1 + v_2)} \\ B &= \frac{(v_1 + v_2)(\bar{h}_1v_1 + \bar{h}_2v_2 + c_1\bar{p}_1v_1^* + c_2\bar{p}_2v_2^*)}{(\bar{h}_1 + \bar{p}_1)v_1 + (\bar{h}_2 + \bar{p}_2)v_2} \\ C &= \frac{\bar{p}_1v_1 + \bar{p}_2v_2 + c_1\bar{h}_1v_1^* + c_2\bar{h}_2v_2^*}{c_1h_1 + c_2h_2 - q} \end{aligned} \quad (25)$$

We can derive ϕ by considering two cases: (1) $v_1 + v_2 \geq c_1v_1^* + c_2v_2^*$ and (2) $v_1 + v_2 < c_1v_1^* + c_2v_2^*$. Consider the case $v_1 + v_2 \geq c_1v_1^* + c_2v_2^*$. We can obtain the result that $R(v_1, v_2) \leq R(v_1^*, v_2^*)$ only when one of the following set of conditions is met:

- (i) $X \in [0, c_1v_1^* + c_2v_2^*]$ and $X \in [0, A]$
- (ii) $X \in (c_1v_1^* + c_2v_2^*, v_1 + v_2]$ and $X \in [c_1v_1^* + c_2v_2^*, B]$
- (iii) $X \in (v_1 + v_2, \infty)$ and $\bar{p}_1v_1 + \bar{p}_2v_2 \leq c_1\bar{p}_1v_1^* + c_2\bar{p}_2v_2^*$

Similarly, for the case $v_1 + v_2 < c_1v_1^* + c_2v_2^*$ one of the following conditions needs to be satisfied:

- (i) $X \in [0, v_1 + v_2]$ and $X \in [0, A]$
- (ii) $X \in (v_1 + v_2, c_1v_1^* + c_2v_2^*]$ and $X \in [C, \infty)$
- (iii) $X \in (c_1v_1^* + c_2v_2^*, \infty)$ and $\bar{p}_1v_1 + \bar{p}_2v_2 \leq c_1\bar{p}_1v_1^* + c_2\bar{p}_2v_2^*$

From the above results, for $\bar{p}_1v_1 + \bar{p}_2v_2 \leq c_1\bar{p}_1v_1^* + c_2\bar{p}_2v_2^*$, we can obtain:

$$\phi(v_1, v_2) = \begin{cases} F[\max\{\min(A, c_1v_1^* + c_2v_2^*), 0\}] + 1 - F[v_1 + v_2] \\ + F[\max\{\min(v_1 + v_2, B), c_1v_1^* + c_2v_2^*\}] - F[c_1v_1^* + c_2v_2^*] & \text{if } v_1 + v_2 \geq c_1v_1^* + c_2v_2^* \\ F[\max\{\min(A, v_1 + v_2), 0\}] + 1 \\ - F[\max\{\min(c_1v_1^* + c_2v_2^*, C), v_1 + v_1\}] & \text{if } v_1 + v_2 < c_1v_1^* + c_2v_2^* \end{cases} \quad (26)$$

Otherwise,

$$\phi(v_1, v_2) = \begin{cases} F[\max\{\min(A, c_1 v_1^* + c_2 v_2^*), 0\}] \\ + F[\max\{\min(v_1 + v_2, B), c_1 v_1^* + c_2 v_2^*\}] - F[c_1 v_1^* + c_2 v_2^*] & \text{if } v_1 + v_2 \geq c_1 v_1^* + c_2 v_2^* \\ F[\max\{\min(A, v_1 + v_2), 0\}] \\ + F[c_1 v_1^* + c_2 v_2^*] - F[\max\{\min(c_1 v_1^* + c_2 v_2^*, C), v_1 + v_2\}] & \text{if } v_1 + v_2 < c_1 v_1^* + c_2 v_2^* \end{cases} \quad (27)$$

Likewise the single PPV advertisement case, the publisher needs to make a contract (v_1, v_2) by solving the following problem:

$$\max_{v_1, v_2} \mathbb{E}[R(v_1, v_2)] = \sum_{i=1}^2 \bar{p}_i v_i + q\mu - \sum_{i=1}^2 (\bar{h}_i + \bar{p}_i) v_i F\left[\sum_{i=1}^2 v_i\right] \\ + \frac{\sum_{i=1}^2 (\bar{h}_i + \bar{p}_i) v_i}{\sum_{i=1}^2 v_i} \int_0^{\sum_{i=1}^2 v_i} x f(x) dx \quad (28)$$

$$\text{subject to } \phi(v_1, v_2) \leq \gamma \quad (29)$$

where ϕ is defined by (26) and (27).

As we have noted before, the convexity of the objective function (28) cannot be shown. However, with the following lemma, we may be able to find the optimal solution.

Lemma 5. If either v_1 or v_2 is constant, then $\mathbb{E}[R(v_1, v_2)]$ is strictly concave with respect to the other variable.

Let us define

$$R_0 = \mathbb{E}[R(v_1, v_2)|_{v_1=0, v_2=0}] = q\mu \quad (30)$$

Now, suppose v_1 is constant and by Lemma 5, there exists a \tilde{v}_2 such that:

$$\tilde{v}_2(v_1) = \arg \max_{v_2} \mathbb{E}[R(v_1, v_2)] := \{\tilde{v}_2 | \forall v_2 : \mathbb{E}[R(v_1, v_2)] < \mathbb{E}[R(v_1, \tilde{v}_2)]\} \quad (31)$$

$\tilde{v}_1(v_2)$ is defined in similar way. About the boundary of the feasible space, we propose the following:

Proposition 3. If $\bar{v}_2 = v_2 : \bar{v}_2 > \tilde{v}_2$ and $\mathbb{E}[R(0, \bar{v}_2)] \leq R_0$ where \tilde{v}_1 is defined by equation (31) and \bar{v}_1 is defined in similar way, the feasible space for the problem (28) lies within $0 < v_1 < \bar{v}_1$ and $0 < v_2 < \bar{v}_2$.

Based on Lemma 5 and Proposition 3, and on the fact that v_1, v_2 are, in fact, integers, we may propose the following algorithm to obtain the solution to problem (28):

Step 0: Find \bar{v}_1 and \bar{v}_2 as defined in Proposition 3. Set $R^* = R_0$, $(v_1^\dagger, v_2^\dagger) = (0, 0)$ and initial $v_1 = 0$. Go to step 1.

Step 1: If $v_1 = \bar{v}_1$, Stop. The optimal solution is $(v_1^\dagger, v_2^\dagger)$ with objective value R^* . Otherwise, go to step 2.

Step 2: Using any line search algorithm, find $\tilde{v}_2(v_1)$ and calculate $\tilde{R}(v_1) = \mathbb{E}[R(v_1, \tilde{v}_2(v_1))]$. If $\tilde{R} \leq R^*$, set $v_1 = v_1 + 1$ go to step 1. Otherwise, go to step 3.

- Step 3: Find the largest integer $v_2^u \in [0, \tilde{v}_2]$ such that $\phi_u(v_1, v_2^u) \leq \gamma$ and calculate $R_u = \mathbb{E}[R(v_1, v_2^u)]$. If no such v_2^u exists, set $R_u = -\infty$. Go to step 4.
- Step 4: Find the smallest integer $v_2^l \in [\tilde{v}_2, \bar{v}_2)$ such that $\phi_l(v_1, v_2^l) \leq \gamma$ and calculate $R_l = \mathbb{E}[\dot{R}(v_1, v_2^l)]$. If no such v_2^l exists, set $R_l = -\infty$. Go to step 5.
- Step 5: Find $\mathbb{R} = \max\{R_u, R_l, R^*\}$. If $R_u = \mathbb{R}$, set $R^* = R_u$ and $(v_1^\dagger, v_2^\dagger) = (v_1, v_2^u)$. If $R_l = \mathbb{R}$, set $R^* = R_l$ and $(v_1^\dagger, v_2^\dagger) = (v_1, v_2^l)$. Set $v_1 = v_1 + 1$ and go to step 1.

Solution of problem (22) can be obtained in similar way.

5 Numerical Results

In this section, we consider the numerical values of the parameters to obtain the results for (22) in single advertisement and (28) in two advertisements. For single advertisement case, we consider a single set of values of the parameters (Table 2). For two advertisements cases, we consider three different sets of parameter values as the result of the unconstrained problem depends on the relation between the terms $\bar{p}_1/\bar{p}_2, \bar{h}_1/\bar{h}_2, \int_0^{v_1^*} xf(x)dx / \int_0^{v_2^*} xf(x)dx$ and $(CR_1 - CR_{12})/(CR_2 - CR_{12})$ (see Table 1). The three sets of parameter values are given in table 3. For set (i), we have $\bar{p}_1/\bar{p}_2 > \bar{h}_1/\bar{h}_2$. The risk-neutral solution for this set of parameter values results in Case 2 ($v_1^* > 0, v_2^* = 0$). For set (ii) and (iii), we have $\bar{p}_1/\bar{p}_2 < \bar{h}_1/\bar{h}_2$. However, for set (ii), we have $\int_0^{v_1^*} xf(x)dx / \int_0^{v_2^*} xf(x)dx \geq (CR_1 - CR_{12})/(CR_2 - CR_{12})$. The risk-neutral solution for parameter set (ii) also results in case 2. On the other hand, for set (iii), we get $\int_0^{v_1^*} xf(x)dx / \int_0^{v_2^*} xf(x)dx < (CR_1 - CR_{12})/(CR_2 - CR_{12})$. The risk-neutral solution for parameter set (iii) results in case 3 ($v_1^* = 0, v_2^* > 0$).

μ	1000
p	1
h	1.5
q	0.75

Table 2: Parameter values for single PPV advertisement

Set	p_1	p_2	h_1	h_2	q
i	1.6	1.5	2.75	2.25	0.25
ii	1.55	1.5	2.7	2.65	0.25
iii	1.85	1.5	3.5	2.65	0.25

Table 3: Parameter values for two PPV advertisements

5.1 Single PPV Advertisement

For this case, we consider values of the parameters specified by Table 2. The page-view follows gamma distribution with shape factor $\alpha = 20$ and scale factor $\beta = 50$ so that the mean page-view $\mu = 1000$. Figure 1 shows that as $v \rightarrow v^*$, the value of γ decreases toward $\gamma_{\min} = \min\left[\frac{\bar{p}}{\bar{p}+\bar{h}}, \frac{\bar{h}}{\bar{p}+\bar{h}}\right]$. This also shows that as the contract number v deviates from v^* , the probability of generating more revenue than *Risk Neutral Strategy* decreases. We consider same gamma distribution to find the value of optimum solution for different values of γ . Figure 2 shows that as the value of γ increases from γ_{\min} , the optimum contract decreases. Also, note that for $\gamma < \gamma_{\min}$, the problem is infeasible i.e. the publisher cannot be completely risk averse. Similarly, there is a

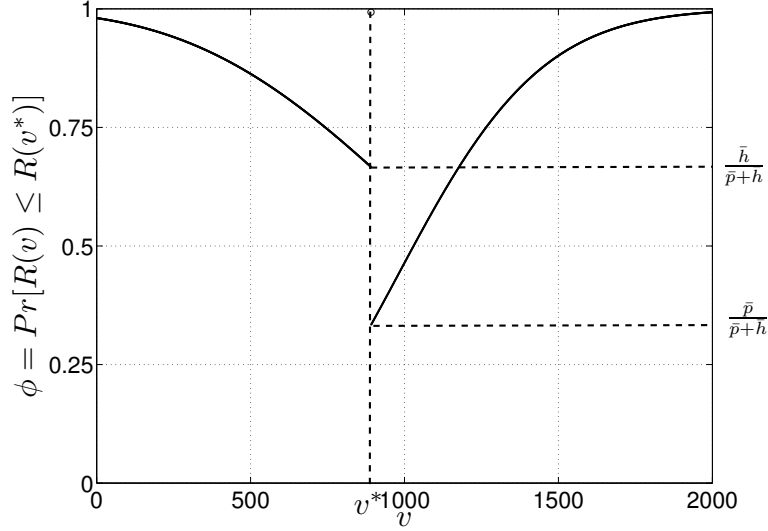


Figure 1: The probability (ϕ) that $R(v) \leq R(v^*)$

limit on the value of γ that specifies maximum risk the publisher may take in order in generate more revenue than *Risk Neutral Strategy*.

5.2 Two PPV Advertisements

We use three sets of parameters values for considering the two PPV advertisements (Table 3). We consider same page-view distribution i.e., gamma distribution with shape factor $\alpha = 20$ and scale factor $\beta = 50$ so that the mean page-view $\mu = 1000$. For set (i), we have $\bar{p}_1/\bar{p}_2 > \bar{h}_1/\bar{h}_2$. Figure 3 shows the optimal solution as γ changes. Note that as PPV 1 advertisement has higher relative CPM and lower relative penalty, optimal contract involves PPV 1 advertisement only. Similar to Single PPV advertisement case, for $\gamma < \gamma_{\min}$, there is no feasible solution where $\gamma_{\min} = \min \left[\frac{\bar{p}_1}{\bar{p}_1 + \bar{h}_1}, \frac{\bar{h}_1}{\bar{p}_1 + \bar{h}_1} \right]$. Also, there is a limit on the value of γ that specifies the maximum risk the publisher may take in order in generate more revenue than *Risk Neutral Strategy*.

For set (ii), we have $\bar{p}_1/\bar{p}_2 < \bar{h}_1/\bar{h}_2$ and $\int_0^{v_1^*} x f(x) dx / \int_0^{v_2^*} x f(x) dx \geq (CR_1 - CR_{12}) / (CR_2 - CR_{12})$. The optimal solution for case 2, as γ changes, is shown in Figure 4. Again, we have $\gamma_{\min} = \min \left[\frac{\bar{p}_1}{\bar{p}_1 + \bar{h}_1}, \frac{\bar{h}_1}{\bar{p}_1 + \bar{h}_1} \right]$ to determine the feasibility of the problem. However, for low values of γ , when the publisher is risk-averse, the optimal contract is mixed contract. This mixed comprises of a with major share of PPV 1 advertisement and a small amount of PPV 2 advertisement. Similar to earlier cases, there is a limit on the value of γ that specifies the maximum risk the publisher may take in order in generate more revenue than *Risk Neutral Strategy*.

For set (iii), we have $\bar{p}_1/\bar{p}_2 < \bar{h}_1/\bar{h}_2$ and $\int_0^{v_1^*} x f(x) dx / \int_0^{v_2^*} x f(x) dx < (CR_1 - CR_{12}) / (CR_2 - CR_{12})$. In this case, contrary to earlier cases, *Risk Neutral Strategy* involves choosing PPV 2 advertisement. For this, we have $\gamma_{\min} = \min \left[\frac{\bar{p}_2}{\bar{p}_2 + \bar{h}_2}, \frac{\bar{h}_2}{\bar{p}_2 + \bar{h}_2} \right]$. The publisher while risk-averse should choose PPV 2 advertisement (Figure 5). For lower values of γ , the contract sizes changes

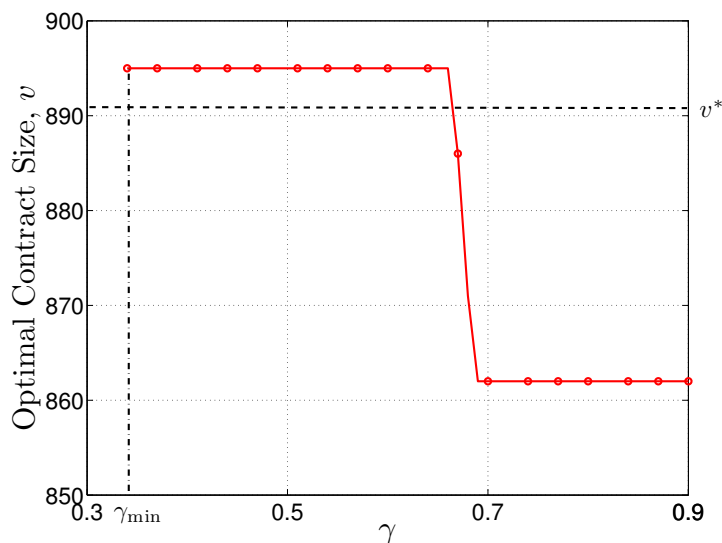


Figure 2: Optimal contract size (v) as γ changes

in the similar way of previous two cases. As the more the publisher becomes risk-seeking, the more of PPV 1 advertisement she will be willing to choose, there is no limit of γ that specifies the maximum risk the publisher may take. Instead, there is a value of γ , beyond which the mixed contract strategy involves choosing more of PPV 1 advertisement as the publisher becomes more risk-seeking.

5.3 Insights on Mixed Contract

We assume that $p_1 > p_2$ and $h_1 > h_2$. This means PPV 1 advertisement is *high-risk high-return* and PPV 2 advertisement is *low-risk low-return*. For parameter set (ii), we find a mixed contract may be optimal when the publisher is highly risk-averse (Figure 4). Note that for lower values of γ , the publisher uses a mixed contract where PPV 1 advertisement is near risk-neutral contract size with a very small PPV 2 advertisement. In this way, she minimizes risk through a mix of *high-risk high-return* and *low-risk low-return* advertisements. As she becomes more risk-seeking, we find she eventually selects *high-risk high-return* PPV 1 advertisement only. For parameter set (iii), we find a mixed contract may be optimal while the publisher is risk-seeking. Note that in this case, PPV 1 advertisement has considerably higher risk. While the publisher is risk-averse (the value of γ is low), the publisher minimizes risk by choosing *low-risk low-return* PPV 2 advertisement only and avoid the risk of paying high penalty (Figure 5). But, as she becomes risk-seeking, she chooses a mixed contract of both *high-risk high-return* PPV 1 advertisement and *low-risk low-return* PPV 2 advertisement. This will give her higher expected revenue along with high risk.

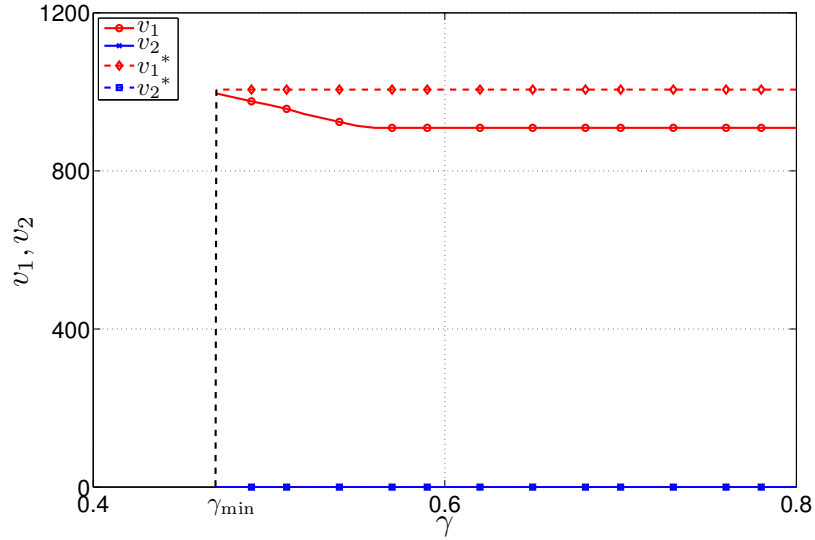


Figure 3: Optimal contract size for parameter set (i) as γ changes.

6 Concluding Remarks

Advertising in the Internet uses different types of display and pricing mechanism. Accordingly, there are many new and unresolved decision problems in online advertising. The problem of finding optimum contract for single and multiple (two) display advertisements is considered here. In display advertisement, revenue is generated through impression based pricing which uses pay-per-view (PPV) as pricing basis. We have provided and analyzed stochastic optimization formulations for optimal contract problems in online advertising. For the risk-neutral publisher, we first considered the single contract problem (the aggregate model). We have shown that it is equivalent to the well-known newsvendor problem, despite the fact that uncertainty comes from the exactly opposite source (capacity vs. demand). With this equivalence, the rich theory of the newsvendor problem will find a new application in online advertising. We also considered an optimization problem with two available contracts. Very importantly, we have shown that a mixed contract is never optimal if the publisher is risk-neutral; hence we only need to consider a single contract problem.

For single PPV advertisement, we consider a risk neutral strategy based on the expected page-view and find solution of the problem so that the revenue is higher than that generated through risk-neutral strategy with some probability. We study how the solution changes as the level of risk changes. We performed a similar study with two PPV advertisements. We consider risk-neutral strategies which consider only one advertisement based on parameter settings and tried to obtain solution so that the revenue generated is higher than that of risk-neutral strategy. We find the solutions for different risk levels. Our specific findings are as follows:

1. The publisher cannot be completely risk averse.
2. As we increase the probability at which optimal strategy would generate less revenue than risk-neutral strategy, the total revenue increases.

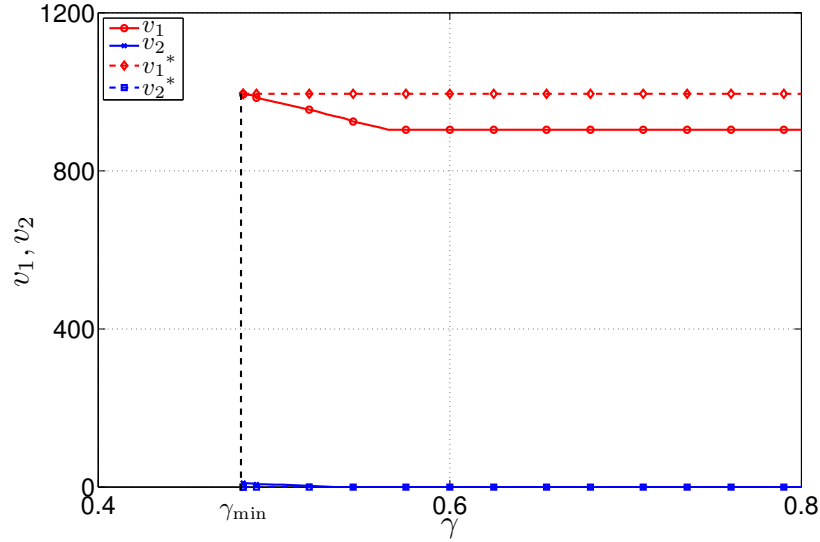


Figure 4: Optimal contract size for parameter set (ii) as γ changes.

3. For single as well as two PPV advertisements, the value of cut-off probability, γ_{\min} depends on parameter set.
4. The nature of solution for different risk levels changes depending on the relation between different parameters.
5. Even though mixed contract is never optimal when the publisher is risk-neutral, we find that mixed-contract may be optimal for risk-averse publisher.
6. As the level of risk increases, the expected revenue for the publisher increases.

We have provided a guideline for the publisher about making contact as well as the display sequence of the advertisements. Depending on the risk the publisher willing to take, we also provide the optimum contract number of advertisements for the publisher. The case of multiple (more than two) PPV advertisements is left for future research.

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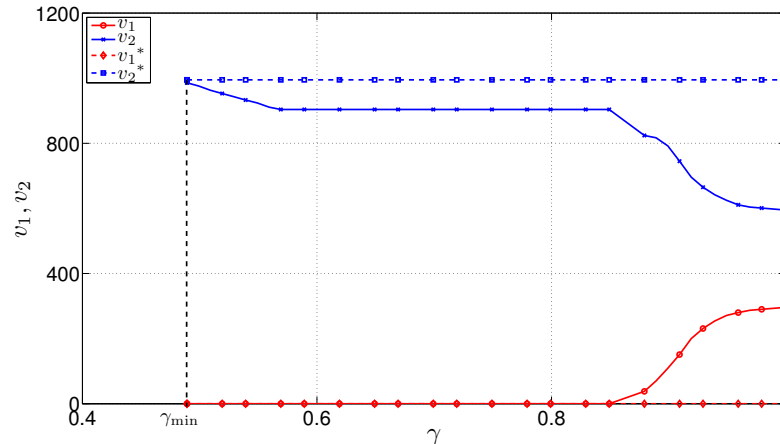


Figure 5: Optimal contract size for parameter set (iii) as γ changes.

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Appendix - Proofs and Derivations

Proof of Proposition 1

In the optimal contract problem (2), we have

$$R(v) = pv - h \max\{v - X, 0\} + q \max\{X - v, 0\}$$

By letting $p = \pi - c$, $h = \pi - s$, $q = -l$ and $X = D$, we obtain

$$\begin{aligned} R(v) &= (\pi - c)v - (\pi - s) \max\{v - D, 0\} - l \max\{D - v, 0\} \\ &= \pi v - cv - \pi \max\{v - D, 0\} + s \max\{v - D, 0\} - l \max\{D - v, 0\} \\ &= \pi \min\{v, D\} - cv + s \max\{v - D, 0\} - l \max\{D - v, 0\} \end{aligned}$$

which is equivalent to the newsvendor's profit function $Z(v)$ in (4). Hence proof.

Proof of Lemma 1

We have

$$\frac{\partial J}{\partial v_1} = \bar{p}_1 - (\bar{h}_1 + \bar{p}_1)F(v_1 + v_2) + \frac{\{(\bar{h}_1 + \bar{p}_1) - (\bar{h}_2 + \bar{p}_2)\}v_2}{(v_1 + v_2)^2} \int_0^{v_1+v_2} xf(x)dx \quad (32)$$

$$\frac{\partial J}{\partial v_2} = \bar{p}_2 - (\bar{h}_2 + \bar{p}_2)F(v_1 + v_2) - \frac{\{(\bar{h}_1 + \bar{p}_1) - (\bar{h}_2 + \bar{p}_2)\}v_1}{(v_1 + v_2)^2} \int_0^{v_1+v_2} xf(x)dx \quad (33)$$

where $F(\cdot)$ is the cdf of X , $f(\cdot)$ is the pdf of X , $\bar{p}_i = p_i - q$ and $\bar{h}_i = h_i - p_i$. The Karush-Kuhn-Tucker (KKT) conditions for (9) are:

$$\lambda_1 = -\frac{\partial J}{\partial v_1} \geq 0 \quad (34)$$

$$\lambda_2 = -\frac{\partial J}{\partial v_2} \geq 0 \quad (35)$$

and

$$\lambda_1 v_1 = 0 \quad (36)$$

$$\lambda_2 v_2 = 0 \quad (37)$$

Without a convexity of J , these conditions only give candidates for local optimum. From (36) and (37) we have

$$\bar{p}_1 v_1 - (\bar{h}_1 + \bar{p}_1)v_1 F(v_1 + v_2) + \frac{\{(\bar{h}_1 + \bar{p}_1) - (\bar{h}_2 + \bar{p}_2)\}v_1 v_2}{(v_1 + v_2)^2} \int_0^{v_1+v_2} xf(x)dx = 0 \quad (38)$$

$$\bar{p}_2 v_2 - (\bar{h}_2 + \bar{p}_2)v_2 F(v_1 + v_2) - \frac{\{(\bar{h}_1 + \bar{p}_1) - (\bar{h}_2 + \bar{p}_2)\}v_1 v_2}{(v_1 + v_2)^2} \int_0^{v_1+v_2} xf(x)dx = 0 \quad (39)$$

By adding (38) and (39), we obtain:

$$F(v_1 + v_2) = \frac{\bar{p}_1 v_1 + \bar{p}_2 v_2}{(\bar{h}_1 + \bar{p}_1)v_1 + (\bar{h}_2 + \bar{p}_2)v_2} \quad (40)$$

Proof of Lemma 2

Let us assume $v_1 > 0$ and $v_2 > 0$ for the KKT conditions. In the complementarity conditions (36) and (37), let us suppose $v_1 > 0$ and $v_2 > 0$. Then, we must have $\frac{\partial J}{\partial v_1} = 0$ and $\frac{\partial J}{\partial v_2} = 0$. Then we have $\frac{\partial J}{\partial v_1} - \frac{\partial J}{\partial v_2} = 0$. That is,

$$(\bar{p}_1 - \bar{p}_2) - \{(\bar{h}_1 + \bar{p}_1) - (\bar{h}_2 + \bar{p}_2)\}F(k) + \frac{\{(\bar{h}_1 + \bar{p}_1) - (\bar{h}_2 + \bar{p}_2)\}k}{k^2} \int_0^k x f(x) dx = 0$$

where $k = v_1 + v_2$. This leads to:

$$F(k) - \frac{1}{k} \int_0^k x f(x) dx = \frac{\bar{p}_1 - \bar{p}_2}{(\bar{h}_1 + \bar{p}_1) - (\bar{h}_2 + \bar{p}_2)} = CR_{12}$$

which can be written as

$$g(k) = F(k) - M(k) = CR_{12} \quad (41)$$

We note that $g(k)$ is a strictly increasing function of k for all $k > 0$, since its first-order derivative is

$$g'(k) = \frac{1}{k^2} \int_0^k x f(x) dx > 0$$

Therefore, if the solution to (41) exists, it is unique.

Using $\frac{\partial J}{\partial v_1} = 0$, $F(k) - M(k) = CR_{12}$ and $v_1 + v_2 = k$, we obtain v_2 as a function of k :

$$\begin{aligned} v_2 &= \frac{(\bar{h}_1 + \bar{p}_1)F(k) - \bar{p}_1}{\frac{\Delta}{k}M(k)} \\ &= \left(\frac{CR_2 - CR_{12}}{CR_2 - CR_1} \right) \left(\frac{F(k) - CR_1}{F(k) - CR_{12}} \right) k \end{aligned}$$

Similarly, using $\frac{\partial J}{\partial v_2} = 0$, $F(k) - M(k) = CR_{12}$ and $v_1 + v_2 = k$, we obtain

$$v_1 = \left(\frac{CR_1 - CR_{12}}{CR_2 - CR_1} \right) \left(\frac{CR_2 - F(k)}{F(k) - CR_{12}} \right) k$$

To ensure $v_1 > 0$ and $v_2 > 0$, $F(k)$ needs to satisfy certain conditions. In particular, if $\bar{h}_1/\bar{h}_2 > \bar{p}_1/\bar{p}_2$, then $CR_1 < F(k) < CR_2$; if $\bar{h}_1/\bar{h}_2 < \bar{p}_1/\bar{p}_2$, then $CR_2 < F(k) < CR_1$, by equation (11).

Proof of Lemma 3

In the complementarity conditions (36) and (37), let us suppose $v_1 > 0$ and $v_2 = 0$. Then, we must have $\frac{\partial J}{\partial v_1} = 0$ and $\lambda_2 = -\frac{\partial J}{\partial v_2} \geq 0$. That is,

$$\frac{\partial J}{\partial v_1} = \bar{p}_1 - (\bar{h}_1 + \bar{p}_1)F(v_1) = 0$$

Therefore we obtain $F(v_1) = CR_1$. Also we have

$$\frac{\partial J}{\partial v_2} = \bar{p}_2 - (\bar{h}_2 + \bar{p}_2)F(v_1) - \Delta M(v_1) \leq 0$$

Dividing the both sides by $(\bar{h}_2 + \bar{p}_2)$, we obtain

$$CR_2 - CR_1 - \frac{\Delta}{\bar{h}_2 + \bar{p}_2} M(v_1) \leq 0$$

or

$$M(v_1) \geq (CR_2 - CR_1) \frac{\bar{h}_2 + \bar{p}_2}{\Delta} = CR_1 - CR_{12}$$

Proof of Lemma 4

In the complementarity conditions (36) and (37), let us suppose $v_1 = 0$ and $v_2 > 0$. Then, we must have $\lambda_1 = -\frac{\partial J}{\partial v_1} \geq 0$ and $\frac{\partial J}{\partial v_2} = 0$. From these two conditions, we obtain the following results as similar to the proof of Lemma 3:

$$\begin{aligned} F(v_2) &= CR_2 \\ M(v_2) &\leq CR_2 - CR_{12} \end{aligned}$$

Proof of Proposition 2

When $v_1 > 0$ and $v_2 > 0$ satisfy the KKT conditions (Case 1), then $k = v_1 + v_2$ is uniquely determined. Therefore, (v_1, v_2) is also uniquely determined. Let us now consider a point $(u_1, u_2) = (v_1 + \delta, v_2 - \delta)$ for a constant δ . We note $u_1 + u_2 = k$. Therefore we have $\mathbb{E}[R(v_1, v_2)] = \mathbb{E}[R(u_1, u_2)]$.

However, we know (u_1, u_2) is not a local optimum (it does not satisfy the KKT conditions). That is, there exists a point $(w_1, w_2) \in \mathcal{N}_{\varepsilon_1}(v_1, v_2)$ such that $\mathbb{E}[R(w_1, w_2)] > \mathbb{E}[R(u_1, u_2)]$ for all $\varepsilon_1 > 0$, where $\mathcal{N}_{\varepsilon}$ denotes the ε -neighborhood. By making $|\delta|$ and ε_1 arbitrarily small, we can make $(w_1, w_2) \in \mathcal{N}_{\varepsilon_2}(v_1, v_2)$ for any $\varepsilon_2 > 0$ and $\mathbb{E}[R(w_1, w_2)] > \mathbb{E}[R(v_1, v_2)]$. Therefore (v_1, v_2) is not a local optimum.

Proof of Lemma 5

Consider v_2 is constant. Differentiating $\mathbb{E}[R(v_1, v_2)]$ with respect to v_1 , we obtain:

$$\frac{\partial \mathbb{E}[R(v_1, v_2)]}{\partial v_1} = \bar{p}_1 - (\bar{h}_1 + \bar{p}_1)F(v_1 + v_2) + \frac{\{(\bar{h}_1 + \bar{p}_1) - (\bar{h}_2 + \bar{p}_2)\}v_2}{(v_1 + v_2)^2} \int_0^{v_1+v_2} xf(x)dX$$

Differentiating again, we get

$$\frac{\partial^2 \mathbb{E}[R(v_1, v_2)]}{\partial v_1^2} = - \left[\frac{f(v_1 + v_2)}{v_1 + v_2} [(\bar{h}_1 + \bar{p}_1)v_1 + (\bar{h}_2 + \bar{p}_2)v_2] + \frac{2(h_1 - h_2)v_2}{(v_1 + v_2)^2} \int_0^{v_1+v_2} xf(x)dX \right] < 0$$

Similarly, when v_1 is constant, we can prove that $\frac{\partial^2 \mathbb{E}[R(v_1, v_2)]}{\partial v_2^2} \leq 0$.

Proof of Proposition 3

Suppose v_1 is constant. Since $\mathbb{E}[R(v_1, v_2)]$ is concave with respect to v_2 by Lemma 5, we have

$$\left. \frac{\partial \mathbb{E}[R(v_1, v_2)]}{\partial v_2} \right|_{\bar{v}_2 > \bar{v}_2} < 0$$

Therefore, $\mathbb{E}[R(v_1, v_2)]$ is decreasing for $v_2 > \tilde{v}_2$.

For case 1 and 2 of Table 1, the risk neutral solution is $\left(F^{-1}\left\{\frac{p_1-q}{h_1-q}\right\}, 0\right)$. Therefore,

$$\begin{aligned}\mathbb{E}\left[R\left(F^{-1}\left\{\frac{p_1-q}{h_1-q}\right\}, 0\right)\right] &= (p_1 - q)v_1 + q\mu - (h_1 - q)v_1 \frac{p_1 - q}{h_1 - q} + (h_1 - q) \int_0^{F^{-1}\left\{\frac{p_1-q}{h_1-q}\right\}} x f(x) dx \\ &= q\mu + (h_1 - q) \int_0^{F^{-1}\left\{\frac{p_1-q}{h_1-q}\right\}} x f(x) dx > R_0\end{aligned}$$

Similarly, for case 3 of Table 1, we have

$$\mathbb{E}\left[R\left(0, F^{-1}\left\{\frac{p_2-q}{h_2-q}\right\}\right)\right] = q\mu + (h_2 - q) \int_0^{F^{-1}\left\{\frac{p_2-q}{h_2-q}\right\}} x f(x) dx > R_0$$

In both cases, we have $\mathbb{E}[R(v_1, \bar{v}_2)] < R_0 < \mathbb{E}[R(v_1^*, v_2^*)]$ and Constraint (29) is never satisfied. Therefore, for each v_1 fixed, any v_2 such that $v_2 > \tilde{v}_2(v_1)$ and $\mathbb{E}[R(v_1, \tilde{v}_2(v_1))] < R_0$ is not a solution to problem (28). Similar result holds vice versa. Now, we have

$$\begin{aligned}\mathbb{E}[R(0, \bar{v}_2)] &= q\mu + \left[(p_2 - q)\bar{v}_2 - (h_2 - q)\bar{v}_2 F[\bar{v}_2] + (h_2 - q) \int_0^{F^{-1}[\bar{v}_2]} x f(x) dx \right] \\ &= R_0 + \left[(p_2 - q)\bar{v}_2 - (h_2 - q)\bar{v}_2 F[\bar{v}_2] + (h_2 - q) \int_0^{F^{-1}[\bar{v}_2]} x f(x) dx \right] \leq R_0\end{aligned}$$

Therefore, $(p_2 - q)\bar{v}_2 - (h_2 - q)\bar{v}_2 F[\bar{v}_2] + (h_2 - q) \int_0^{F^{-1}[\bar{v}_2]} x f(x) dx \leq 0$. As $\mathbb{E}[R(0, \bar{v}_2)]$ is decreasing for $v_2 \geq \bar{v}_2$, any increase in total contract size will further decrease the $\mathbb{E}[R]$, i.e. $\mathbb{E}[R(v_1, \bar{v}_2)] < \mathbb{E}[R(0, \bar{v}_2)]$.