Optimal Contract-Sizing in Online Display Advertising for Publishers with Regret Considerations

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Abstract

In this paper, we study optimal contract problems for online display advertisements with pay-per-view pricing scheme. We first provide and analyze a single contract model, which is shown to be equivalent to the newsvendor problem. We then consider a stochastic optimization problem with two different advertisements and show that a contract to display both of them is not optimal. However, we show that a contract to display of both advertisements may be optimal when we consider the publisher's regret. We consider a chance constraint for the publisher's regret and provide numerical experiments that illustrate the change of optimal strategy for different probability levels.

Keywords: Newsboy problem; Operations management; Risk.

1 Introduction

Online advertising has already become a dominant ad-medium and is continuously gaining market share. In the United States, with 2011 revenue of \$31.74 billion, online advertising is now marking increasing significance to marketers and consumers (Interactive Advertising Bureau, 2012a). Online advertising in the US is expected to grow eight times faster than the overall market and is already the second largest behind direct marketing with a \$9.3 billion revenue in the third quarter of 2012 (Interactive Advertising Bureau, 2012b). According to Interactive Advertising Bureau (2012a), search and display are the major formats of online advertising having 81.3% market share. Search advertisement appears as a search result in a web page loaded in response to a user keyword(s) search request. Display advertisement, on the other hand, is displayed in web pages that are requested for different purposes. Rather than instantaneous response, display advertising, in general, focuses in achieving memory. Even though both search and display format will exist, reports suggest

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that marketeers are becoming more interested in display advertising. Instead of making a keyword search, more users are making online purchases in response to display advertising (eMarketer, 2004). Such gain in market share is attributed to the use of vast potential for technological innovation and creativity in providing display advertisement (Interactive Advertising Bureau, 2010). From 2009 to 2010, the market share for display increased for 34% to 36% resulting about 5% relative gain while the market for search remains the same (Interactive Advertising Bureau, 2010). As a result, display will remain one of the leading modes of online advertising.

There are several pricing methods for online advertising services. Among those, pay-per-view (PPV) and pay-per-click (PPC) are dominant in the market. In particular, PPV is popular in display advertising and PPC is popular in search-based keyword advertising. For example, most news websites like BBC use PPV schemes (BBC, 2013) for displaying ad banners, while search engines like Google mainly use PPC schemes (Google, 2013) for search keywords. In this paper, we focus on PPV schemes for display advertising.

There are two major sources of uncertainty in online advertising. First, the publisher does not have any prior information about the number page-views i.e. how many times the publisher's page will be accessed by users. The second uncertainty is the number of clicks that is the total number of click on the advertising. In this study, we consider the case that the publisher has decided to host display advertisements only, which use pay-per-view pricing schemes, or the nature of the problem suggests so. We will examine how to make contracts optimally while the number of page-views is uncertain. With this end in view, we first consider that there is only one type of PPV advertisement available and the publisher needs to determine contract size. Next, we consider the case where there are two types of PPV advertisement available. Here, the publisher needs to determine whether she should go for both advertisements (multiple contracts), and what should be the optimal contract size. In particular, we study the optimal contract problem for single as well as two PPV advertisements. We also make a similar study of optimal contract decision problem when the publisher considers regret. In particular, we model the regret of the publisher with the probability that the revenue is less than a certain number.

In a perfect competitive market, everyone generally accepts the market price so that it would generate reasonable demand for one's product. Similarly, the publisher is considered a price-taker in this study. Therefore, she needs to determine the size of the contract that maximizes the revenue. The optimal contract size should be the upper limit on the promise she should be making. In case the demand at any pricing structure is less than the revenue-maximizing contract, the publisher should make the contract equal to the demand and then try to decide on contract size of the next best option. The effect of price on demand and optimizing the contract size as well as price can be considered in an oligopoly. In online advertising, Ahmed and Kwon (2012) has studied this problem where each publisher needs to optimize the price and contract size considering the effect of price on demand. In this study, we assumed that the publisher, in a perfect competition, has unlimited number of display advertising available. Therefore, there is no limit on the size of contract the publisher can make.

Regarding online advertising, there are limited studies reported in the literature of operations research. Mangani (2004) addressed optimal decision making between cost-per-impression (CPM) and cost-per-click (CPC) advertisements when the publishers are price-takers in the display advertisement market and later Fjell (2009) revisited the same problem. Both Mangani (2004) and Fjell (2009) used the concept of elasticity to reach the conclusions. Kwon (2011) also studied a capacity allocation problem between CPM and CPC advertisements with stochastic page-view and click-through rate (CTR) and provided a stochastic optimization formulation. Kumar and Sethi (2009) considered a dynamic pricing problem considering subscription and advertising. Roels and Fridgeirsdottir (2009) studied a dynamic optimal customer selection and display scheduling problem considering only CPM contracts. Fjell (2010) first considered the problem for a publisher who has the market power to set the price. However, to the knowledge of the authors, no study on optimal contract problems considering risk/regret has been reported. Deane and Agarwal (2012) study a display scheduling problem for display advertisements.

In this paper, we provide formulations for contract-sizing problems of online display ad publishers. The modeling framework is similar to Ahmed and Kwon (2012) for which price is the main decision variable, while in this paper the size of contract is the primary decision. Our models also inherit modeling components of Roels and Fridgeirsdottir (2009), in the sense that we consider revenue opportunities from third-party risk-free ad networks and lost revenue opportunities from a revenue optimization perspective. This paper further extends the previous modeling frameworks to consider regrets of ad publishers. We find that a regret-averse ad publisher may consider two different contract types under PPV pricing schemes, while a neutral ad publisher would not consider two different contract types.

Our study of regret-averse ad publishers is closely related to risk management in stochastic inventory problems. The main difference is that *supply* is stochastic in our online advertising problem, while *demand* is stochastic in most inventory problems. A good number of studies in stochastic inventory management consider Value-at-Risk (VaR) as the measure of risk. Özler et al. (2009) studied single, two and multi-product newsvendor problem incorporating VaR risk measure. The exact formulation of the problem for single and two products case has been studied. They also found the results for two products for the case of correlated demand. An approximation method has been proposed to obtain the results for a multi-products case. Luciano et al. (2003) also studied multi-period static inventory models with VaR being the risk measure. Zhou et al. (2008) proposed an optimal-order model to consider multi-product inventory problem with Conditional Value-at-Risk (CVaR) constraints. The model is simulated for the case of a newsvendor problem and they find the solution bound is fully consistent with the decision maker's intuition on return-risk decision-making. The return-CVaR model is found to be more flexible than the classical model. Yang et al. (2008) considered the risk of a newsvendor with limited capacity. Both the downside risk measure and CVaR risk measure has been used here to optimize the model. Tapiero (2005) also addressed stochastic inventory control problem with VaR approach, and showed that VaR approach is justified by a disappointment criterion and provides its applications to inventory management.

Agrawal and Seshadri (2000) studied how price and order quantity is impacted by uncertainty and risk aversion. They found that risk-averse retailers distort their pricing decision in different ways depending on the impact of price on the distribution of demand. Other examples for newsvendor problems with risk considerations include the studies of Keren and Pliskin (2006), Van Mieghem (2007), Choi and Ruszczynski (2008), Chen et al. (2009) and Wang and Webster (2009), Arcelus et al. (2012), and Jörnsten et al. (2012).

In this work, *display* of an advertisement refers to loading a web-page containing the advertisement in response to an online user request to load that page. When a page containing an advertisement is displayed, we count that as a single *impression*. The number of *page-views* is a random variable specifying how many times a publisher's web-page containing an advertisement is requested by visitors. The term *capacity* is used synonymously with the number of page-views, in the perspective of web publishers for online advertising services. After the period of consideration is over, the total number of display of an advertisement is termed as *realized display* of that advertisement. For displaying only one advertisement, realized display and realized number of page-views are same. If the publisher displays more than one advertisement, she needs to decide on allocation and *display rule*. The publisher may display an advertisement first until the contracted number of displays, and then start displaying another advertisement. We call this display rule a sequential display rule. On the other hand, the publisher may display advertisements proportionally. That is, the publisher displays advertisements in a mixed sequence of various advertisements. We call this display rule a proportional display rule. In this study, we consider proportional display rule only.

The paper is organized as follows. Section 2 provides the optimal contract problem for a single contract, and provides the equivalence of the optimal contract problem to the well-known newsvendor problem. Then, Section 3 considers an optimal contract problem with two display advertisements contracts and shows that making multiple contracts is not optimal. Section 4 formulates the problems for single and two PPV advertisements considering a chance constraint for regret, which is followed by numerical examples in Section 5. The numerical results show that making a contract to display both advertisements may be optimal when publisher is willing to take a high chance of paying large penalty to generate more revenue. Finally, Section 6 concludes the study.

2 Optimal Contract Sizing Problem: Single PPV Advertisement

In this section, we will develop a model for the optimal contract-sizing problem in online advertising. In particular, we will consider a contract-sizing problem for displaying online advertisements with pay-per-view (PPV) pricing scheme. Payment for PPV is dependent on cost-per-impression (CPM), hence the revenue of a publisher depends on how many impressions she can be able to display.

When we consider a publisher's optimal contract sizing problem, two types of advertisements have PPV pricing scheme: one is advertisements by direct contract with the publisher, and the other is so-called network advertisements (Roels and Fridgeirsdottir, 2009). The former type, henceforth termed as PPV advertisements, involves a contract that specifies how many times the advertisements will be displayed within a certain time-frame. Network advertisements are sourced through third parties or network agencies that make the contact with the advertisers. These network advertisements are available for a flat rate lower price. The publisher generally tends to display network advertisements to fulfill the unused capacity after displaying the PPV advertisements.

When the publisher makes a direct contract with an advertiser to display a PPV advertisement, she promises a certain number of displays. For any reason, when the publisher cannot display the contracted number, there is a penalty to the publisher for such under-delievery (The Digital Marketing Glossary, 2012; Mostagir, 2010; Roels and Fridgeirsdottir, 2009; Bhalgat et al., 2012). Such a penalty usually refers to the lost opportunity of the publisher by carrying over the unrealized displays to the next time period. The penalty may also refer to direct refunds to the advertiser and the loss of goodwill. However, for network advertisements, there is no such promise on the number of displays.

As such a penalty is an important component of the operation of online advertising systems, the number of page-views for the future time period is critical to the publisher. A page-view is defined as a request to load a single page on a visitor's screen. The total number of page-views is definitely subject to uncertainty, i.e., the publisher can only have a forecast on this value. Similar to stochastic inventory problems (Schweitzer and Cachon, 2000), we can make several assumptions about the uncertainty of page-view. Let X ($X \geq 0$) denote stochastic page-views during the time period of interest and let μ be its mean. Let $F(.)$ be the distribution function of page-views and $f(.)$ the density function. For simplicity, we assume $F(.)$ is continuous, differentiable and strictly increasing. Further, we assume that the publisher (decision maker) has an unbiased forecast of the page-view distribution and knows $F(.)$. In addition, we assume $E[X] > 0$, to avoid trivial cases.

In this section, we assume the publisher has only one web page to display and only one advertisement can be displayed at a time. Also, the publisher will either make a single contract, or make contracts with many advertisers who are homogeneous (same CPM and penalty). We provide a formulation to consider the single contract problem or the aggregate contract problem. Our objective is to determine the optimal number of displays that should be promised to the advertiser(s). Let v denote the aggregate number of displays to promise (decision variable). We use p and h for the homogeneous CPM and penalty, respectively. Although CPM is the cost for a thousand impressions, we use p for one impression, for simplicity. Then, with the page-view X, the revenue of the publisher is:

$$R(v) = pv - h \max\{(v - X), 0\} + q \max\{(X - v), 0\} \tag{1}$$

where q is the unit revenue from network advertisements. If there are extra page-views after fulfilling all contracts, the publisher uses those extra capacities for the network advertisement. When we have the distribution information of X, the optimization problem

$$\max_{v} \mathbb{E}[R(v)] \tag{2}$$

is a simple stochastic optimization problem. Its solution is easily obtained as:

$$v^* = F^{-1}\left(\frac{p-q}{h-q}\right) = F^{-1}\left(\frac{\overline{p}}{\overline{p}+\overline{h}}\right) \tag{3}$$

where $F(\cdot)$ is the cumulative distribution function of X ($X \geq 0$). Also, we have $\overline{p} = p - q$ and $\overline{h} = h - p$. We make a reasonable assumption that we have $h > p > q$ so that $\overline{p}$ and $\overline{h}$ are positive.

We can show the relationships between the optimal contract problem (2) and the well-known newsvendor problem (Petruzzi and Dada, 1999). The main difference between the two problems is that demand information is uncertain in the newsvendor problem, while capacity (page-views) is uncertain in the optimal contract problem (2).

We use the following notation: order quantity Q, stochastic demand D with CDF $G(\cdot)$, unit purchase cost c, unit sales price π, unit salvage value s, and unit lost opportunity cost l. We assume $\pi > c > s$. The newsvendor problem can be formulated as following:

$$\max_{Q} \mathbb{E}[Z(Q)] = \mathbb{E}[\pi \min\{Q, D\} - cQ + s \max\{(Q-D), 0\} - l \max\{(D-Q), 0\}] \tag{4}$$

and its solution is

$$Q^* = G^{-1}\left(\frac{\pi+l-c}{\pi+l-s}\right) = G^{-1}\left(\frac{\overline{\pi}}{\overline{\pi}+\overline{c}}\right) \tag{5}$$

where $\overline{\pi} = \pi + l - c$ and $\overline{c} = c - s$.

The optimal contract sizing problem (2) has a very similar structure to the newsvendor problem (4), and in fact they are equivalent. A proof of the following proposition is provided in Appendix. All the other proofs are also provided in Appendix.

Proposition 1. *The optimal contract sizing problem (2) is a newsvendor problem with the sales price π, the purchase cost $\pi - p$, the salvage value $\pi - h$, the lost opportunity cost $-q$ and the demand X, for any arbitrary π such that $\pi > h > p > q$.*

For a publisher, the optimal contract size sets the upper limit so that she can decide the number of display of advertisement she should commit. It is possible that there are many advertisers who want to make contracts with the publisher, so that the total number of display demand is higher than the publisher's optimal contract size. In this case, the publisher should accept the optimal contract size only, rejecting any demand beyond it. On the other hand, if there is demand less than the optimal contract size, accepting all of them is the best strategy for the publisher.

3 Optimal Contract Sizing Problem: Two PPV Advertisements

The single contract sizing problem (2) with the homogeneity assumption is simple and equivalent to the well-known newsvendor problem. However, when we take into account the case of more than one heterogeneous PPV advertisements, the model for the optimal contract-sizing problem becomes quickly complicated. In this section, we consider an optimal contract-sizing problem with two PPV advertisements that are involved with the same page. Like the single PPV advertisement problem, we assume that only one advertisement can be displayed at a time. Therefore, it is important to decide the display order of the two PPV advertisements. If the publisher has enough page-views so that no PPV contract invokes penalty, the display order is not important. On the other hand, if she does not have enough number of page-views to display the contracted numbers, the display order becomes important, because the amount of penalty she pays depends on the display order. If she decides to display the advertisement with highest CPM (p_i) first, the other advertiser may find this unfair which may lead to loss of future business. Also, if the advertisement with higher penalty (h_i) is displayed later, the publisher might end up paying too much as penalty if the page-view is too low. To avoid this, if the publisher chooses to display advertisement with higher penalty first, she might lose potential earning if page-view fails to meet the contract obligation marginally. At least for the sake of fairness, we consider proportional display rule (d_1, d_2), so that the *realized* number of displays of advertisement i becomes:

$$\min\left\{v_i, \frac{d_i}{d_1+d_2}X\right\}$$

where v_i is the contract size (decision variable) of displays for PPV advertisement i and X is the (stochastic) number of page-views. Since the above relation uses ratio, the unit of d_i is not important. The display rule (d_1, d_2) would be implemented in sequence. For example, if $d_1 = 3$ and $d_2 = 5$, any consecutive 8 page-views will constitute three displays of advertisement 1 and five of advertisement 2. With display rule (d_1, d_2), the revenue becomes:

$$\begin{aligned} R(v_1, v_2) = p_1 v_1 + p_2 v_2 &- h_1 \max\left[v_1 - \max\left\{\frac{d_1}{d_1+d_2}X, X - v_2\right\}, 0\right] \\ &- h_2 \max\left[v_2 - \max\left\{\frac{d_2}{d_1+d_2}X, X - v_1\right\}, 0\right] + q\max[X-(v_1+v_2), 0] \end{aligned} \quad (6)$$

Here, $\frac{d_1}{d_1+d_2}X$ is fraction of page-view capacity allocated to advertisement 1 and $(X - v_2)$ is the remaining capacity after displaying advertisement 2. The publisher has to pay penalty for advertisement 1 if $v_1 > \max\left\{\frac{d_1}{d_1+d_2}X, X - v_2\right\}$. Using display rule (d_1, d_2), if the publisher finished displaying advertisement 2, then we have $X - v_2 = \max\left\{\frac{d_1}{d_1+d_2}X, X - v_2\right\}$. Otherwise, we have $\frac{d_1}{d_1+d_2}X = \max\left\{\frac{d_1}{d_1+d_2}X, X - v_2\right\}$. For example, if we have $(v_1, v_2) = (600, 120)$ and $X = 500$. For $(d_1, d_2) = (2, 1)$, for the first 360 page-view, 240 will be advertisement 1 while 120 will be advertisement 2. The remaining capacity of 140 will be allocated to advertisement 1. After this, the publisher has to pay penalty for failing to provide the remaining 220 display of advertisement 1. We can explain the case of advertisement 2 in a similar manner.

We may assume any values for d_i i.e. $d_i = v_i$, $d_i = p_i$, $d_i = p_i + h_i$, or $d_i = h_i$. In order to be fair to both advertisers, we consider $d_i = v_i$ in this study. With $d_i = v_i$, if the realized number of page-views is 10% less than what is expected, each advertisement is displayed 10% less times than the contracted number of display regardless the price or penalty. When $d_i = v_i$, objective function

(6) reduces to:

$$R(v_1, v_2) = p_1 v_1 + p_2 v_2 - h_1 \max\left[v_1 - \frac{v_1}{v_1+v_2}X, 0\right] \\ - h_2 \max\left[v_2 - \frac{v_2}{v_1+v_2}X, 0\right] + q\max[X - (v_1+v_2), 0] \quad (7)$$

The expected value of (7) is:

$$\begin{aligned} \mathbb{E}[R(v_1, v_2)] &= p_1 v_1 + p_2 v_2 + q\mu - \{(h_1 v_1 + h_2 v_2) - q(v_1+v_2)\}F(v_1+v_2) \\ &\quad - q(v_1+v_2) + \left(\frac{h_1 v_1}{v_1+v_2} + \frac{h_2 v_2}{v_1+v_2} - q\right)\int_0^{v_1+v_2} xf(x)dx \\ &= q\mu + \bar{p}_1 v_1 + \bar{p}_2 v_2 - \{(\bar{h}_1 + \bar{p}_1)v_1 + (\bar{h}_2 + \bar{p}_2)v_2\}F(v_1+v_2) \\ &\quad + \left(\frac{(\bar{h}_1 + \bar{p}_1)v_1 + (\bar{h}_2 + \bar{p}_2)v_2}{v_1+v_2}\right)\int_0^{v_1+v_2} xf(x)dx \end{aligned} \quad (8)$$

where $\bar{p}_i = p_i - q$ and $\bar{h}_i = h_i - p_i$.

Finally, the optimal contract-sizing problem for the two PPV advertisements becomes:

$$\max_{v_1, v_2 \geq 0} J = \max_{v_1, v_2 \geq 0} \mathbb{E}[R(v_1, v_2)] \quad (9)$$

Note that, unlike in the aggregate optimal contract-sizing problem, for multiple PPV cases, there is no equivalence to the newsvendor problem. Without loss of generality, we assume $p_1 > p_2$. In this study, we only consider $h_1 > h_2$. If $p_1 > p_2$ and $h_1 < h_2$, ad 1 is always better than ad 2; therefore, we need not consider such case. We let λ_1 and λ_2 denote the dual variables for the non-negativity constraints.

Considering the Karush-Kuhn-Tucker (KKT) conditions, we will investigate the following cases separately:

	$v_1 > 0$	$v_1 = 0$
$v_2 > 0$	Case 1	Case 3
$v_2 = 0$	Case 2	Case 4

When we find solutions of the KKT conditions in each case, we will have *candidates* for the optimum. Let us assume h_1 and h_2 are not very large, so that Case 4 is never optimal. Also note that, without a convexity of J, KKT conditions only give candidates for local optimal solutions. Before we begin, we first obtain the following necessary condition:

Proposition 2. *The following condition is necessary for an optimal solution to (9):*

$$F(v_1 + v_2) = \frac{\bar{p}_1 v_1 + \bar{p}_2 v_2}{(\bar{h}_1 + \bar{p}_1)v_1 + (\bar{h}_2 + \bar{p}_2)v_2} \quad (10)$$

Proofs to Proposition 2 and other subsequent lemmas and propositions are provided in Appendix. If we compare (10) with (3), we see that the total number of contract $(v_1 + v_2)$ is relevant to a weighted ratio of $\overline{p}_i$ to $\overline{h}_i + \overline{p}_i$. Let us introduce additional notations and definitions:

$$M(y) = \frac{1}{y}\int_0^y xf(x)dx, \qquad g(y) = F(y) - M(y), \qquad \Delta = (\overline{h}_1 + \overline{p}_1) - (\overline{h}_2 + \overline{p}_2),$$
$$CR_1 = \frac{\overline{p}_1}{\overline{h}_1 + \overline{p}_1}, \qquad CR_2 = \frac{\overline{p}_2}{\overline{h}_2 + \overline{p}_2}, \qquad CR_{12} = \frac{\overline{p}_1 - \overline{p}_2}{(\overline{h}_1 + \overline{p}_1) - (\overline{h}_2 + \overline{p}_2)}$$

Note that $\Delta = (h_1 - p_1 + p_1 - q) - (h_2 - p_2 + p_2 - q) = h_1 - h_2 > 0$ by the assumption. Hence we have the following relation:

$$\begin{aligned} CR_{12} < CR_1 < CR_2 \qquad & \text{if} \quad \overline{h}_1/\overline{h}_2 > \overline{p}_1/\overline{p}_2 \\ CR_2 < CR_1 < CR_{12} \qquad & \text{if} \quad \overline{h}_1/\overline{h}_2 < \overline{p}_1/\overline{p}_2 \end{aligned} \tag{11}$$

For each case, we obtain the following result:

Lemma 1 (Case 1: $v_1 > 0$, $v_2 > 0$)**.** *Let k be a solution of the equation $g(k) = CR_{12}$. If such k exists, it is unique. Suppose the following condition holds:*

$$\begin{aligned} CR_1 < F(k) < CR_2 \qquad & \textit{if } \overline{h}_1/\overline{h}_2 > \overline{p}_1/\overline{p}_2 \\ CR_2 < F(k) < CR_1 \qquad & \textit{if } \overline{h}_1/\overline{h}_2 < \overline{p}_1/\overline{p}_2 \end{aligned}$$

Then the following pair:

$$v_1^* = \left(\frac{CR_1 - CR_{12}}{CR_2 - CR_1}\right)\left(\frac{CR_2 - F(k)}{F(k) - CR_{12}}\right)k$$
$$v_2^* = \left(\frac{CR_2 - CR_{12}}{CR_2 - CR_1}\right)\left(\frac{F(k) - CR_1}{F(k) - CR_{12}}\right)k$$

satisfies the KKT conditions. Therefore, it is a unique candidate in Case 1 for local optimum.

The conditions in Lemma 1 identify a candidate point for an optimal solution that satisfiy the necessary condition in Proposition 2. However, we will observe later this is a saddle point in Proposition 3. That is, the (v_1^*, v_2^*)-pair in Lemma 1 is a stationary point, but it is not a local optimum.

Lemma 2 (Case 2: $v_1 > 0$, $v_2 = 0$)**.** *If the following condition is satisfied:*

$$M(\rho_1) \geq CR_1 - CR_{12} \tag{12}$$

where $\rho_1 = F^{-1}(CR_1)$, then $v_1^* = \rho_1$ and $v_2^* = 0$ is a unique KKT candidate for local optimum in Case 2.

Note that when $\overline{h}_1/\overline{h}_2 < \overline{p}_1/\overline{p}_2$, we automatically have $CR_1 - CR_{12} < 0$ by equation (11), therefore, condition (12) is always satisfied. That is, when the advertisement 1 has higher relative revenue $\overline{p}_1/\overline{p}_2$ than penalty $\overline{h}_1/\overline{h}_2$, it is always optimal that the publisher make a contract only with the advertisement 1.

Lemma 3 (Case 3: $v_1 = 0$, $v_2 > 0$)**.** *Let us have ρ_2 such that $F(\rho_2) = CR_2$. If we have*

$$M(\rho_2) \leq CR_2 - CR_{12} \tag{13}$$

then $v_1^* = 0$ and $v_2^* = F^{-1}(CR_2)$ satisfy the KKT conditions; hence a unique candidate for local optimum in Case 3.

We note that when $\overline{h}_1/\overline{h}_2 < \overline{p}_1/\overline{p}_2$, we automatically have $CR_2 - CR_{12} < 0$ by equation (11), therefore, the condition (13) is never satisfied. This is consistent with the discussion of Lemma 2.

As Lemmas 1, 2 and 3 only give candidates for optimum, we need to compare the objective function values to identify the global solution. We obtain the expected revenue at the local solution of Case 1:

$$\begin{aligned}
\mathbb{E}[R]_{\text{Case 1}} &= \overline{p}_1 v_1 + \overline{p}_2 v_2 + q\mu - [(\overline{h}_1 + \overline{p}_1)v_1 + (\overline{h}_2 + \overline{p}_2)v_2]CR_{12} \\
&= q\mu + \frac{\overline{p}_2\overline{h}_1 - \overline{p}_1\overline{h}_2}{\Delta}(v_1 + v_2) \\
&= q\mu + \frac{\overline{p}_2\overline{h}_1 - \overline{p}_1\overline{h}_2}{\Delta}k
\end{aligned} \tag{14}$$

where $g(k) = CR_{12}$. For Cases 2 and 3, we obtain:

$$\begin{aligned}
\mathbb{E}[R]_{\text{Case 2}} &= \overline{p}_1 v_1 + q\mu - (\overline{h}_1 + \overline{p}_1)v_1(F(v_1) - M(v_1)) \\
&= q\mu + (\overline{h}_1 + \overline{p}_1)v_1 M(v_1) \\
&= q\mu + (\overline{h}_1 + \overline{p}_1)\int_0^{F^{-1}(CR_1)} xf(x)dx
\end{aligned} \tag{15}$$

and

$$\mathbb{E}[R]_{\text{Case 3}} = q\mu + (\overline{h}_2 + \overline{p}_2)\int_0^{F^{-1}(CR_2)} xf(x)dx \tag{16}$$

Note that (14) is only dependent on k. Using this fact, we provide the following proposition:

Proposition 3. *The solution from Case 1, as in Lemma 1, is not a local optimum.*

We make an important point: for any case, a contract to display both advertisements ($v_1 > 0, v_2 > 0$) is not optimal. Therefore, either Case 2 or Case 3 (but not both) is optimal. A contract to display both advertisements is optimal only when they are identical, i.e., $p_1 = p_2$ and $h_1 = h_2$. That is, publisher does not need to consider differentiated contracts (Case 1).

The optimal solution can be determined by comparing (15) and (16). When $\overline{h}_1/\overline{h}_2 < \overline{p}_1/\overline{p}_2$, we know Case 3 is never optimal. Therefore, we only consider when $\overline{h}_1/\overline{h}_2 < \overline{p}_1/\overline{p}_2$. Comparing (15)

Parameter Set	Condition 1	Condition 2	Neutral Solution $\{v_1^*, v_2^*\}$
i	$\frac{\overline{p}_1}{\overline{p}_2} \geq \frac{\overline{h}_1}{\overline{h}_2}$		$\left\{F^{-1}\left(\frac{\overline{p}_1}{\overline{p}_1+\overline{h}_1}\right), 0\right\}$
ii	$\frac{\overline{p}_1}{\overline{p}_2} < \frac{\overline{h}_1}{\overline{h}_2}$	$\frac{\int_0^{F^{-1}(CR_1)} xf(x)dx}{\int_0^{F^{-1}(CR_2)} xf(x)dx} > \frac{CR_1 - CR_{12}}{CR_2 - CR_{12}}$	$\left\{F^{-1}\left(\frac{\overline{p}_1}{\overline{p}_1+\overline{h}_1}\right), 0\right\}$
iii	$\frac{\overline{p}_1}{\overline{p}_2} < \frac{\overline{h}_1}{\overline{h}_2}$	$\frac{\int_0^{F^{-1}(CR_1)} xf(x)dx}{\int_0^{F^{-1}(CR_2)} xf(x)dx} = \frac{CR_1 - CR_{12}}{CR_2 - CR_{12}}$	$\left\{F^{-1}\left(\frac{\overline{p}_1}{\overline{p}_1+\overline{h}_1}\right), 0\right\}$ or, $\left\{0, F^{-1}\left(\frac{\overline{p}_2}{\overline{p}_2+\overline{h}_2}\right)\right\}$
iv	$\frac{\overline{p}_1}{\overline{p}_2} < \frac{\overline{h}_1}{\overline{h}_2}$	$\frac{\int_0^{F^{-1}(CR_1)} xf(x)dx}{\int_0^{F^{-1}(CR_2)} xf(x)dx} < \frac{CR_1 - CR_{12}}{CR_2 - CR_{12}}$	$\left\{0, F^{-1}\left(\frac{\overline{p}_2}{\overline{p}_2+\overline{h}_2}\right)\right\}$

Table 1: The optimal solution for different conditions

and (16), we obtain that Case 2 is optimal if

$$\frac{\displaystyle\int_0^{F^{-1}(CR_1)} xf(x)dx}{\displaystyle\int_0^{F^{-1}(CR_2)} xf(x)dx} \geq \frac{CR_1 - CR_{12}}{CR_2 - CR_{12}} \tag{17}$$

On the other hand, Case 3 is optimal if

$$\frac{\displaystyle\int_0^{F^{-1}(CR_1)} xf(x)dx}{\displaystyle\int_0^{F^{-1}(CR_2)} xf(x)dx} \leq \frac{CR_1 - CR_{12}}{CR_2 - CR_{12}} \tag{18}$$

As multiple advertising is not optimal, either Case 2 (${v_1}^* = 0, {v_2}^* > 0$) or Case 3 (${v_1}^* > 0, {v_2}^* = 0$), but not both, is optimal while we have

$$\frac{\displaystyle\int_0^{F^{-1}(CR_1)} xf(x)dx}{\displaystyle\int_0^{F^{-1}(CR_2)} xf(x)dx} = \frac{CR_1 - CR_{12}}{CR_2 - CR_{12}} \tag{19}$$

Table 1 summarizes these results for Case 2 and Case 3 based on the parameters and the resulting conditions obtained from the parameters. Note that for parameter sets (iv), Case 2 (${v_1}^* = 0, {v_2}^* > 0$) is the optimal solution while for parameter sets (i) and (ii), the optimal outcome is always Case 3 (${v_1}^* > 0, {v_2}^* = 0$). For parameter set (iii), either Case 2 or Case 3 is optimal.

In the following section, we will show that a contract to display both advertisements can be optimal when we consider the publisher's preference on 'regret' by introducing a chance constraint.

4 Optimal Strategy Considering the Publisher's Regret

In this section, our objective is to maximize the expected profit subject to a chance constraint, which models the publisher's regret and is defined as the probability of earning less than a specific revenue (R^*) is less than or equal to the threshold probability value γ (Özler et al., 2009). In this study, we select R^* as the expected revenue by a solution to problem (2) or (9), which we will refer to the *Neutral Solution* or *Neutral Strategy*.

4.1 Single PPV advertisement

As we have mentioned, the publisher considers to adopt a strategy based on the neutral contract size, i.e., $v^* = F^{-1}\left\{\bar{p}/(\bar{h}+\bar{p})\right\}$. Accordingly, she makes a contract to display the PPV advertisement v^* number of times. The revenue for this *Neutral Strategy* will be:

$$R(v^*) = pv^* - h\max(v^* - X, 0) + q\max(X - v^*, 0) \tag{20}$$

The publisher will want to choose a contract size other than v^* only if the strategy is expected to generate more revenue than the *Neutral Strategy* with a probability she is willing to accept. In other words, she wants to make a contract v other than v^* such that the realized page-view will generate more revenue than v^* with a specified probability $(1-\gamma)$. Let us consider the probability that a strategy v generates higher revenue than the *Neutral Strategy* with a probability of at least $(1-\gamma)$. In particular, we consider the regret probability ϕ such that:

$$\phi = \Pr[R(v) < R(v^*)] \leq \gamma \tag{21}$$

Here, $R(v)$ and $R(v^*)$ are defined by equations (1) and (20) respectively. We may use any number instead of $R(v^*)$ in (21). However, in this paper, we are interested in the publisher's regret compared to the *Neutral Strategy*, and therefore (21) is relevant to the probability of regret to the *Neutral Strategy*. For a conservative publisher, the value of γ is lower while γ is higher for an aggressive publisher.

The values of ϕ can be found by evaluating the following expression:

$$\begin{aligned} R(v) - R(v^*) = p(v - v^*) - h\{\max(v - X, 0) - \max(v^* - X, 0)\} \\ + q\{\max(X - v, 0) - \max(X - v^*, 0)\} \end{aligned} \tag{22}$$

Considering the range of X, the exact expression for equation (21) becomes (see Appendix for details):

$$\phi(v) = \begin{cases} F\left(\dfrac{\overline{h}v + \overline{p}v^*}{\overline{h} + \overline{p}}\right) & \text{if } v > v^* \\ \\ 1 - F\left(\dfrac{\overline{h}v^* + \overline{p}v}{\overline{h} + \overline{p}}\right) & \text{if } v < v^* \\ \\ 0 & \text{if } v = v^* \end{cases} \tag{23}$$

Note that the value of ϕ decreases as $v \to v^*$ because we can easily verify that:

$$\frac{\partial \phi}{\partial v} < 0 \qquad \text{for all } v < v^*$$
$$\frac{\partial \phi}{\partial v} > 0 \qquad \text{for all } v > v^*$$

Also, we find the cut-off probability:

$$\gamma_{\min} = \min\left\{\lim_{v \to v^{*+}} \phi, \lim_{v \to v^{*-}} \phi\right\} = \min\{F(v^*), 1 - F(v^*)\} = \min\left\{\frac{\overline{p}}{\overline{h} + \overline{p}}, \frac{\overline{h}}{\overline{h} + \overline{p}}\right\}$$

Accordingly, the publisher can never be able to earn more revenue than the *Neutral Strategy* with a probability greater than $(1 - \gamma_{\min})$. Therefore we cannot find a strategy which will generate more expected revenue than the neutral strategy if $\gamma < \gamma_{\min}$. Note that as the parameters p, h, q or $\overline{p}, \overline{h}$ are constant, the value of $\gamma_{\min}$ remains constant irrespective of probability distribution of page-view X.

Consider that the publisher wants to maximize revenue through making a contract v such that the revenue is higher than that generated through *Neutral Strategy* (v^*) with a probability of at least $(1 - \gamma)$ or $\phi \leq \gamma$. The problem can be written in terms of probability density and cumulative distribution function of v as:

$$\max_v \mathbb{E}[R(v)] = \overline{p}v + q\mu - (\overline{h} + v)vF(v) + (\overline{h} + \overline{p}) \int_0^v xf(x)dx \tag{24}$$

subject to

$$\phi(v) \leq \gamma$$

where ϕ is defined by (23).

The problem (24) for single PPV advertisement has concave objective function and piece-wise convex feasible space. In particular, the feasible space has three continuous and convex segments i.e., for $v \in [0, v^*)$, $v = v^*$ or $v > v^*$. Problem (24) can be considered a piece-wise convex program. A piece-wise convex program is a convex program such that the constraint set can be decomposed in a finite number of closed convex sets such that on each of these cells, the objective function can be described by a continuously differentiable convex function (Louveaux, 1978). Accordingly, the optimal solutions to problem (24) can be obtained by finding and comparing solution obtained by considering each concave segment of the feasible space.

4.2 Two PPV advertisements

In the case of two PPV advertisements, we find that the neutral publisher should make a contract to display only one of the advertisements following the guidelines provided in Table 1. In this section, we consider the publisher's regret for the case of two PPV advertisements. Let us first consider the expected revenue from the *Neutral Strategy* which is:

$$R({v_1}^*, {v_2}^*) = \chi_1 p_1 {v_1}^* + \chi_2 p_2 {v_2}^* - \chi_1 h_1 \max({v_1}^* - X, 0) - \chi_2 h_2 \max({v_2}^* - X, 0) \\ + q \max(X - \chi_1 {v_1}^* - \chi_2 {v_2}^*, 0) \quad (25)$$

where χ_i's are binary valued and $\{\chi_1, \chi_2\} = \{1, 0\}$ when the neutral solution, according to Table 1, is ${v_1}^* > 0, {v_2}^* = 0$; and $\{\chi_1, \chi_2\} = \{0, 1\}$ if ${v_1}^* = 0, {v_2}^* > 0$.

Similar to the single PPV advertisement case, we consider the probability ϕ that an arbitrary strategy (v_1, v_2) generates no more revenue than the *Neutral Strategy*:

$$\phi = \Pr[R(v_1, v_2) < R({v_1}^*, {v_2}^*)] \leq \gamma$$

For $\overline{p}_1 v_1 + \overline{p}_2 v_2 \leq \chi_1 \overline{p}_1 {v_1}^* + \chi_2 \overline{p}_2 {v_2}^*$, we have (see Appendix for details):

$$\phi(v_1, v_2) = \begin{cases} F[\max\{\min(A, \chi_1 {v_1}^* + \chi_2 {v_2}^*), 0\}] + 1 - F[v_1 + v_2] \\ +F[\max\{\min(v_1 + v_2, B), \chi_1 {v_1}^* + \chi_2 {v_2}^*\}] - F[\chi_1 {v_1}^* + \chi_2 {v_2}^*] & \text{if } v_1 + v_2 \geq \chi_1 {v_1}^* + \chi_2 {v_2}^* \\ \\ F[\max\{\min(A, v_1 + v_2), 0\}] + 1 \\ -F[\max\{\min(\chi_1 {v_1}^* + \chi_2 {v_2}^*, C), v_1 + v_1\}] & \text{if } v_1 + v_2 < \chi_1 {v_1}^* + \chi_2 {v_2}^* \end{cases} \quad (26)$$

Otherwise,

$$\phi(v_1, v_2) = \begin{cases} F[\max\{\min(A, \chi_1 {v_1}^* + \chi_2 {v_2}^*), 0\}] \\ +F[\max\{\min(v_1 + v_2, B), \chi_1 {v_1}^* + \chi_2 {v_2}^*\}] - F[\chi_1 {v_1}^* + \chi_2 {v_2}^*] & \text{if } v_1 + v_2 \geq \chi_1 {v_1}^* + \chi_2 {v_2}^* \\ \\ F[\max\{\min(A, v_1 + v_2), 0\}] \\ +F[\chi_1 {v_1}^* + \chi_2 {v_2}^*] - F[\max\{\min(\chi_1 {v_1}^* + \chi_2 {v_2}^*, C), v_1 + v_1\}] & \text{if } v_1 + v_2 < \chi_1 {v_1}^* + \chi_2 {v_2}^* \end{cases} \quad (27)$$

where

$$A = \frac{(v_1 + v_2)\left[\overline{h}_1(v_1 - \chi_1 v_1{}^*) + \overline{h}_2(v_2 - \chi_2 v_2{}^*)\right]}{h_1 v_1 + h_2 v_2 - (\chi_1 h_1 + \chi_2 h_2)(v_1 + v_2)}$$
$$B = \frac{(v_1 + v_2)(\overline{h}_1 v_1 + \overline{h}_2 v_2 + \chi_1 \overline{p}_1 v_1{}^* + \chi_2 \overline{p}_2 v_2{}^*)}{(\overline{h}_1 + \overline{p}_1)v_1 + (\overline{h}_2 + \overline{p}_2)v_2}$$
$$C = \frac{\overline{p}_1 v_1 + \overline{p}_2 v_2 + \chi_1 \overline{h}_1 v_1{}^* + \chi_2 \overline{h}_2 v_2{}^*}{\chi_1 h_1 + \chi_2 h_2 - q}$$

Likewise the single PPV advertisement case, the publisher needs to make a contract (v_1, v_2) by solving the following problem:

$$\max_{v_1,v_2} \mathbb{E}[R(v_1, v_2)] = \overline{p}_1 v_1 + \overline{p}_2 v_2 + q\mu - \left\{(\overline{h}_1 + \overline{p}_1)v_1 + (\overline{h}_2 + \overline{p}_2)v_2\right\} F(v_1 + v_2) + \frac{(\overline{h}_1 + \overline{p}_1)v_1 + (\overline{h}_2 + \overline{p}_2)v_2}{(v_1 + v_2)} \int_0^{v_1+v_2} x f(x)dx \quad \text{subject to} \quad \phi(v_1, v_2) \le \gamma \tag{28}$$

where ϕ is defined by (26) and (27).

As we have noted before, the concavity of the objective function of the problem (28) cannot be shown. However, the following lemma can help finding a search procedure to obtain the optimal solution.

Lemma 4. *For a given v_1, $\mathbb{E}[R(v_1, v_2)]$ is strictly concave with respect to v_2. Similarly, for a given v_2, $\mathbb{E}[R(v_1, v_2)]$ is strictly concave with respect to v_1.*

Therefore, as we see in Lemma 4, for a given value of v_1, we can find the optimal solution $(v_2^\dagger)$ using any line search or gradient search procedure and vice versa. We can continue to search for all possible values of v_1 and could obtain the optimal solution $(v_1^\dagger,\, v_2^\dagger)$ by comparison.

Let us define

$$R_0 = \mathbb{E}[\,R(v_1, v_2)|_{v_1=0, v_2=0}] = q\mu \tag{29}$$

According to Lemma 4, there exist $\tilde{v}_1$ and $\tilde{v}_2$ such that:

$$\tilde{v}_1(v_2) = \arg\max_{v_1} \mathbb{E}[R(v_1, v_2)] := \{\tilde{v}_1 | \forall v_1 : \mathbb{E}[R(v_1, v_2)] < \mathbb{E}[R(\tilde{v}_1, v_2)]\} \tag{30}$$
$$\tilde{v}_2(v_1) = \arg\max_{v_2} \mathbb{E}[R(v_1, v_2)] := \{\tilde{v}_2 | \forall v_2 : \mathbb{E}[R(v_1, v_2)] < \mathbb{E}[R(v_1, \tilde{v}_2)]\}$$

About the boundary of the feasible space, we propose the following:

Proposition 4. *If $\bar{v}_1 = v_1 : \bar{v}_1 > \tilde{v}_1$ and $\mathbb{E}[R(\bar{v}_1, 0)] \le R_0$ where $\tilde{v}_1$ is defined by equation (30) and $\bar{v}_2$ is defined similarly, the feasible space for the problem (28) lies within $0 < v_1 < \bar{v}_1$ and $0 < v_2 < \bar{v}_2$.*

From Proposition 4, we can conclude that the optimal solution, if exists, will be within space bounded by $0 < v_1 < \bar{v}_1$ and $0 < v_2 < \bar{v}_2$. Therefore, given that a feasible solution exists, any search within this bounded space is guaranteed to provide the optimal solution. Based on Lemma 4 and Proposition 4, and on the fact that v_1, v_2 are, in fact, integers, we may propose the following algorithm to obtain the solution to problem (28):

Step 0: Find $\bar{v}_1$ and $\bar{v}_2$ as defined in Proposition 4. Set $R^\dagger = R_0$, $(v_1^\dagger, v_2^\dagger) = (0, 0)$ and initial $v_1 = 0$. Go to step 1.

Step 1: If $v_1 = \bar{v}_1$, Stop. The optimal solution is $(v_1^\dagger, v_2^\dagger)$ with objective value $R^\dagger$. Otherwise, go to step 2.

Step 2: Using any line search algorithm, find $\tilde{v}_2(v_1)$ and calculate $\tilde{R}(v_1) = \mathbb{E}[R(v_1, \tilde{v}_2(v_1))]$. If $\tilde{R} \leq R^\dagger$, set $v_1 = v_1 + 1$ go to step 1. Otherwise, go to step 3.

Step 3: Find the largest integer $v_2^u \in [0, \tilde{v}_2]$ such that $\phi_u(v_1, v_2^u) \leq \gamma$ and calculate $R_u = \mathbb{E}[R(v_1, v_2^u]$. If no such v_2^u exists, set $R_u = -\infty$. Go to step 4.

Step 4: Find the smallest integer $v_2^l \in [\tilde{v_2}, \bar{v}_2)$ such that $\phi_l(v_1, v_2^l) \leq \gamma$ and calculate $R_l = \mathbb{E}[\dot{R}(v_1, v_2^l)]$. If no such v_2^l exists, set $R_l = -\infty$. Go to step 5.

Step 5: Find $\mathbb{R} = \max\{R_u, R_l, R^\dagger\}$. If $R_u = \mathbb{R}$, set $R^\dagger = R_u$ and $(v_1^\dagger, v_2^\dagger) = (v_1, v_2^u)$. If $R_l = \mathbb{R}$, set $R^\dagger = R_l$ and $(v_1^\dagger, v_2^\dagger) = (v_1, v_2^l)$. Set $v_1 = v_1 + 1$ and go to step 1.

Solution of problem (24) can be obtained in similar way.

5 Numerical Results

In this section, we consider the numerical values of the parameters to obtain the results for (24) in single advertisement and (28) in two advertisements. For single advertisement case, we consider a single set of values of the parameters (Table 2). For two advertisements cases, we consider four different sets of parameter values as the result of the unconstrained problem depends on the relation between the terms $\overline{p}_1/\overline{p}_2$, $\overline{h}_1/\overline{h}_2$, $\int_0^{v_1^*} xf(x)dx/\int_0^{v_2^*} xf(x)dx$ and $(CR_1 - CR_{12})/(CR_2 - CR_{12})$ (see Table 1). The four sets of parameter values are given in table 3. For set (i), we have $\overline{p}_1/\overline{p}_2 > \overline{h}_1/\overline{h}_2$. The neutral solution for this set of parameter values is Case 2 ($v_1^* > 0, v_2^* = 0$). For set (ii), (iii) and (iv), we have $\overline{p}_1/\overline{p}_2 < \overline{h}_1/\overline{h}_2$. However, for set (ii), we have $\int_0^{v_1^*} xf(x)dx/\int_0^{v_2^*} xf(x)dx > (CR_1 - CR_{12})/(CR_2 - CR_{12})$ and for set (iii), we have $\int_0^{v_1^*} xf(x)dx/\int_0^{v_2^*} xf(x)dx = (CR_1 - CR_{12})/(CR_2 - CR_{12})$. The neutral solution for parameter set (ii) also results in case 2. For parameter set (iii), as we have noted in Table 1, optimal solution is either Case 2 or Case 3. For set (iv), we get $\int_0^{v_1^*} xf(x)dx/\int_0^{v_2^*} xf(xt)dx < (CR_1 - CR_{12})/(CR_2 - CR_{12})$. The neutral solution for parameter set (iv) results in case 3 ($v_1^* = 0, v_2^* > 0$).

α	20
β	50
p	1
h	1.5
q	0.75

Table 2: Parameter values for single PPV ad

Set	p_1	p_2	h_1	h_2	q
i	2.25	1.5	3	2	0.25
ii	1.75	1.5	3	2	0.25
iii	1.61	1.5	3	2	0.25
iv	1.55	1.5	3	2	0.25

Table 3: Parameter values for two PPV ads

5.1 Single PPV Advertisement

For this case, we consider values of the parameters specified by Table 2. The page-view follows gamma distribution with shape factor $\alpha = 20$ and scale factor $\beta = 50$ so that the mean page-view $\mu = 1000$. Figure 1 shows that as $v \to v^*$, the value of γ decreases toward $\gamma_{\min} = \min\left[\frac{\overline{p}}{\overline{p}+\overline{h}}, \frac{\overline{h}}{\overline{p}+\overline{h}}\right]$. This also shows that as the contract number v deviates from v^*, the probability of generating more revenue than the *Neutral Strategy* decreases. We consider same gamma distribution to find the value of optimum solution for different values of γ. Figure 2 shows that as the value of γ increases from $\gamma_{\min}$, the optimum contract size increases. Also, note that for $\gamma < \gamma_{\min}$, the problem is infeasible.

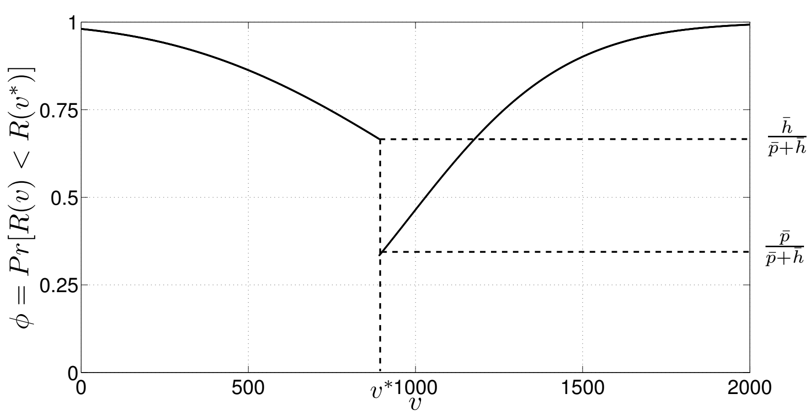

Figure 1: The probability (ϕ) that $R(v) < R(v^*)$

5.2 Two PPV Advertisements

We use four sets of parameters values for considering the two PPV advertisements (Table 3). We consider same page-view distribution i.e., gamma distribution with shape factor $\alpha = 20$ and scale factor $\beta = 50$ so that the mean page-view $\mu = 1000$. For set (i), we have $\overline{p}_1/\overline{p}_2 > \overline{h}_1/\overline{h}_2$. Figure 3 shows the optimal solution as γ changes. Note that as PPV 1 advertisement has higher relative CPM and lower relative penalty, optimal contract involves PPV 1 advertisement only. Similar to Single PPV advertisement case, for $\gamma < \gamma_{\min}$, there is no feasible solution where $\gamma_{\min} = \min\left[\frac{\overline{p}_1}{\overline{p}_1+\overline{h}_1}, \frac{\overline{h}_1}{\overline{p}_1+\overline{h}_1}\right]$.

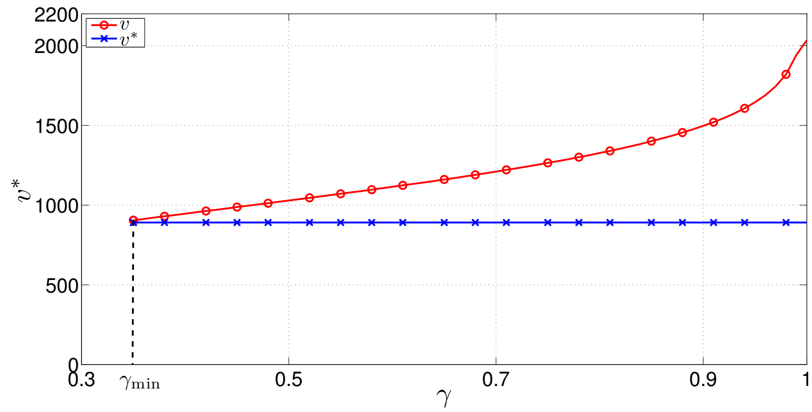

Figure 2: Optimal contract size (v) as γ changes

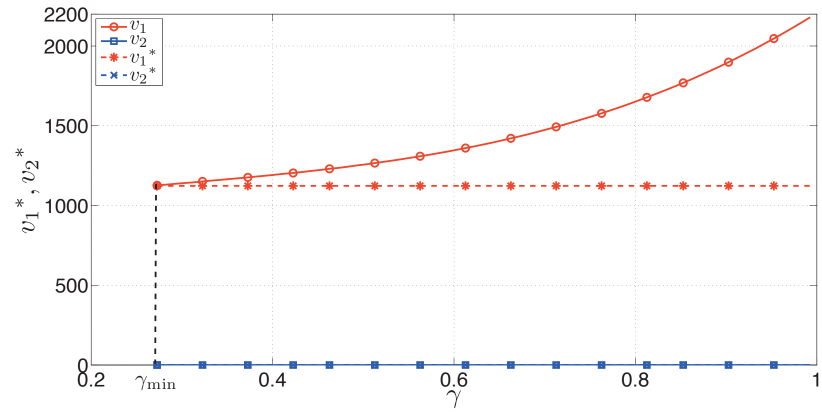

Figure 3: Optimal contract size for parameter set (i) as γ changes.

For set (ii), we have $\overline{p}_1/\overline{p}_2 < \overline{h}_1/\overline{h}_2$ and $\int_0^{v_1^*} xf(x)dx / \int_0^{v_2^*} xf(x)dx > (CR_1 - CR_{12})/(CR_2 - CR_{12})$. The optimal solution for case 2, as γ changes, is shown in Figure 4. Again, we have $\gamma_{\min} = \min\left[\frac{\overline{p}_1}{\overline{p}_1+\overline{h}_1}, \frac{\overline{h}_1}{\overline{p}_1+\overline{h}_1}\right]$ to determine the feasibility of the problem. However, for low values of γ, the optimal contract is to have multiple contracts. This optimal contract is comprised of a major share of PPV 1 advertisement and a small amount of PPV 2 advertisement.

For set (iii), we have $\overline{p}_1/\overline{p}_2 < \overline{h}_1/\overline{h}_2$ and $\int_0^{v_1^*} xf(x)dx / \int_0^{v_2^*} xf(x)dx = (CR_1 - CR_{12})/(CR_2 - CR_{12})$. As the neutral solution for set of parameters is either Case 2 ($v_1^* > 0, v_2^* = 0$) or Case 3 ($v_1^* = 0, v_2^* > 0$), we have numerical experiment for both Case 2 (Figure 5) and Case 3 (Figure 6). As we see in Figure 5, when we consider Case 2 ($v_1^* > 0, v_2^* = 0$), the optimal solution for various γ level is similar to parameter set (ii). When we consider Case 3 ($v_1^* = 0, v_2^* > 0$), the optimal solution (Figure 5) for different γ level is similar to parameter set (iv) which will be discussed next.

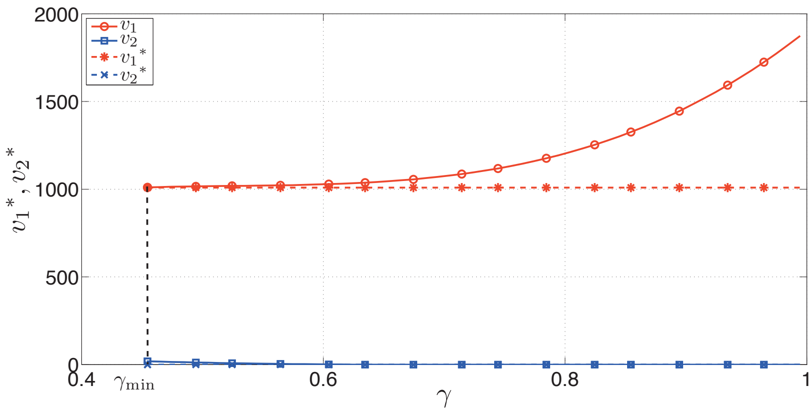

Figure 4: Optimal contract size for parameter set (ii) as γ changes.

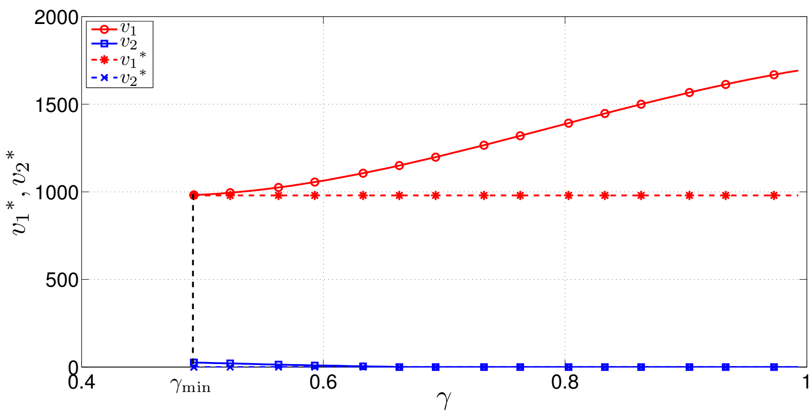

Figure 5: Optimal contract size for parameter set (iii) as γ changes when $(v_1^* > 0, v_2^* = 0)$.

For set (iv), we have $\overline{p}_1/\overline{p}_2 < \overline{h}_1/\overline{h}_2$ and $\int_0^{v_1^*} xf(x)dx/\int_0^{v_2^*} xf(x)dx < (CR_1 - CR_{12})/(CR_2 - CR_{12})$. In this case, contrary to earlier cases, *Neutral Strategy* involves choosing PPV 2 advertisement. For this, we have $\gamma_{\min} = \min\left[\frac{\overline{p}_2}{\overline{p}_2+\overline{h}_2}, \frac{\overline{h}_2}{\overline{p}_2+\overline{h}_2}\right]$. While the value of γ is low, the publisher should choose PPV 2 advertisement (Figure 7). For lower values of γ, the optimal contract size of PPV 2 advertisement changes similar to the way the contract size PPV 1 advertisement changes in the previous two cases. The more the publisher is more aggressive, the more of PPV 1 advertisement she will be willing to choose. Instead, there is a value of γ, beyond which the contract strategy involves choosing more of PPV 1 advertisement.

5.3 Insights on Multiple Contracts

We assume that $p_1 > p_2$ and $h_1 > h_2$. This means PPV 1 advertisement is *high-penalty high-revenue* and PPV 2 advertisement is *low-penalty low-revenue*. For parameter set (ii), we find making multiple contracts may be optimal when the publisher's regret-probability parameter value

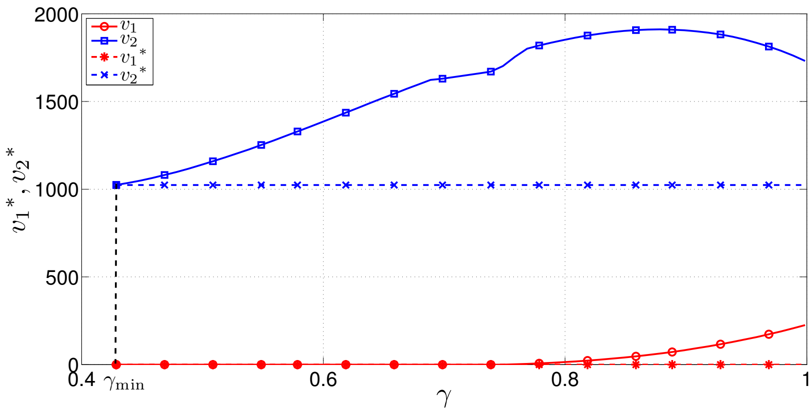

Figure 6: Optimal contract size for parameter set (iii) as γ changes when ($v_1^* = 0, v_2^* > 0$).

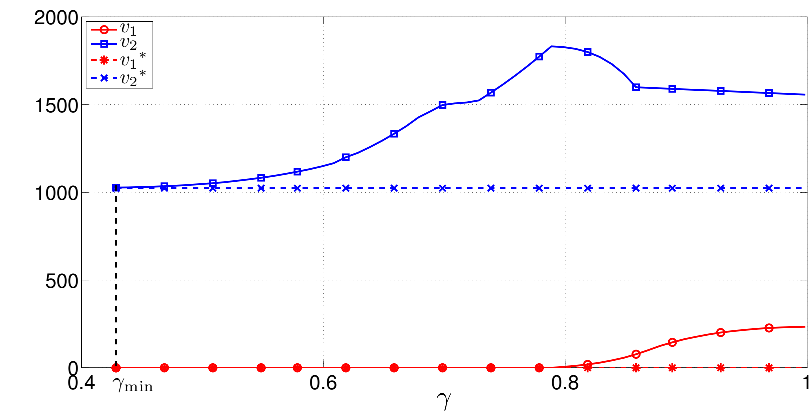

Figure 7: Optimal contract size for parameter set (iv) as γ changes.

γ is lower (Figure 4). Note that for lower values of γ, the publisher make multiple contracts where PPV 1 advertisement is near the neutral contract size with a very small PPV 2 advertisement. In this way, she minimizes regret through a mix of *high-penalty high-revenue* and *low-penalty low-revenue* advertisements. As she becomes more aggressive, we find she eventually selects *high-penalty high-revenue* PPV 1 advertisement only. We find similar result for parameter set (iii) when the neutral solution is considered to be $v_1^* > 0$, $v_2^* = 0$. For parameter set (iv), we find making multiple contracts may be optimal while the publisher is more aggressive. Note that in this case, PPV 1 advertisement has considerably higher risk. While the value of γ is low, the publisher minimizes regret by choosing *low-penalty low-revenue* PPV 2 advertisement only and thus avoids the risk of paying high penalty (Figure 7). But, as γ becomes larger, she chooses to make multiple contracts of both *high-penalty high-revenue* PPV 1 advertisement and *low-penalty low-revenue* PPV 2 advertisement. Parameter set (iii) also shows similar result when $v_1^* = 0$, $v_2^* > 0$ is considered.

6 Concluding Remarks

Advertising in the Internet uses different types of display and pricing mechanism. Accordingly, there are many new and unresolved decision problems in online advertising. The problem of finding optimum contract size for single and two display advertisements is considered here. In display advertising, revenue is generated through impression based pricing which uses pay-per-view (PPV) pricing basis. We have provided and analyzed stochastic optimization formulations for optimal contract sizing problems in online display advertising. We first considered the single contract-sizing problem (the aggregate model). We have shown that it is equivalent to the well-known newsvendor problem, despite the fact that uncertainty comes from the exactly opposite source (capacity vs. demand). With this equivalence, the rich theory of the newsvendor problem will find a new application in online advertising. We also considered an optimization problem when there are two types of display advertisements available. Very importantly, we have shown that a contract to display both advertisements is never optimal if the publisher is neutral and there is no other constraint; hence we only need to consider a single contract-sizing problem. We called such a solution the *neutral solution*.

For single PPV advertisement, we consider a chance constraint that models the publisher's regret. In particular, we considered the probability of the revenue being less than the revenue from the neutral solution is less than or equal to a certain probability threshold. We have studied how the solution changes as the probability threshold changes. We performed a similar study with two PPV advertisements. Our specific findings are as follows:

1. The neutral publisher should display only one advertisement when there is no constraint.

2. It may be optimal for the publisher to display both advertisement when a chance constraint for regret is considered.

3. For single as well as two PPV advertisements, there exists a $\gamma_{\min}$ for each parameter set such that for $\gamma < \gamma_{\min}$, the publisher's problem becomes infeasible.

4. The nature of solution for different probability levels changes depending on the relation between different parameters.

We have provided a guideline for the publisher about making contact as well as the display sequence of the advertisements. Depending on the probability, we also provide the optimum contract number of advertisements for the publisher. The case of multiple (more than two) PPV advertisements is left for future research.

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Appendix - Proofs and Derivations

Proof of Proposition 1

In the optimal contract problem (2), we have

$$R(v) = pv - h\max\{v - X, 0\} + q\max\{X - v, 0\}$$

By letting $p = \pi - c$, $h = \pi - s$, $q = -l$ and $X = D$, we obtain

$$\begin{aligned} R(v) &= (\pi - c)v - (\pi - s)\max\{v - D, 0\} - l\max\{D - v, 0\} \\ &= \pi v - cv - \pi\max\{v - D, 0\} + s\max\{v - D, 0\} - l\max\{D - v, 0\} \\ &= \pi\min\{v, D\} - cv + s\max\{v - D, 0\} - l\max\{D - v, 0\} \end{aligned}$$

which is equivalent to the newsvendor's profit function $Z(v)$ in (4). Hence proof.

Proof of Proposition 2

We have

$$\frac{\partial J}{\partial v_1} = \overline{p}_1 - (\overline{h}_1 + \overline{p}_1)F(v_1 + v_2) + \frac{\{(\overline{h}_1 + \overline{p}_1) - (\overline{h}_2 + \overline{p}_2)\}v_2}{(v_1 + v_2)^2}\int_0^{v_1+v_2} xf(x)dx \tag{31}$$

$$\frac{\partial J}{\partial v_2} = \overline{p}_2 - (\overline{h}_2 + \overline{p}_2)F(v_1 + v_2) - \frac{\{(\overline{h}_1 + \overline{p}_1) - (\overline{h}_2 + \overline{p}_2)\}v_1}{(v_1 + v_2)^2}\int_0^{v_1+v_2} xf(x)dx \tag{32}$$

where $F(\cdot)$ is the cdf of X, $f(\cdot)$ is the pdf of X, $\overline{p}_i = p_i - q$ and $\overline{h}_i = h_i - p_i$. The Karush-Kuhn-Tucker (KKT) conditions for (9) are:

$$\lambda_1 = -\frac{\partial J}{\partial v_1} \geq 0 \tag{33}$$

$$\lambda_2 = -\frac{\partial J}{\partial v_2} \geq 0 \tag{34}$$

and

$$\lambda_1 v_1 = 0 \tag{35}$$
$$\lambda_2 v_2 = 0 \tag{36}$$

Without a convexity of J, these conditions only give candidates for local optimum. From (35) and (36) we have

$$\overline{p}_1 v_1 - (\overline{h}_1 + \overline{p}_1) v_1 F(v_1 + v_2) + \frac{\{(\overline{h}_1 + \overline{p}_1) - (\overline{h}_2 + \overline{p}_2)\} v_1 v_2}{(v_1 + v_2)^2} \int_0^{v_1+v_2} x f(x) dx = 0 \tag{37}$$

$$\overline{p}_2 v_2 - (\overline{h}_2 + \overline{p}_2) v_2 F(v_1 + v_2) - \frac{\{(\overline{h}_1 + \overline{p}_1) - (\overline{h}_2 + \overline{p}_2)\} v_1 v_2}{(v_1 + v_2)^2} \int_0^{v_1+v_2} x f(x) dx = 0 \tag{38}$$

By adding (37) and (38), we obtain:

$$F(v_1 + v_2) = \frac{\overline{p}_1 v_1 + \overline{p}_2 v_2}{(\overline{h}_1 + \overline{p}_1) v_1 + (\overline{h}_2 + \overline{p}_2) v_2} \tag{39}$$

Proof of Lemma 1

Let us assume $v_1 > 0$ and $v_2 > 0$ for the KKT conditions. In the complementarity conditions (35) and (36), let us suppose $v_1 > 0$ and $v_2 > 0$. Then, we must have $\frac{\partial J}{\partial v_1} = 0$ and $\frac{\partial J}{\partial v_2} = 0$. Then we have $\frac{\partial J}{\partial v_1} - \frac{\partial J}{\partial v_2} = 0$. That is,

$$(\overline{p}_1 - \overline{p}_2) - \{(\overline{h}_1 + \overline{p}_1) - (\overline{h}_2 + \overline{p}_2)\} F(k) + \frac{\{(\overline{h}_1 + \overline{p}_1) - (\overline{h}_2 + \overline{p}_2)\} k}{k^2} \int_0^k x f(x) dx = 0$$

where $k = v_1 + v_2$. This leads to:

$$F(k) - \frac{1}{k} \int_0^k x f(x) dx = \frac{\overline{p}_1 - \overline{p}_2}{(\overline{h}_1 + \overline{p}_1) - (\overline{h}_2 + \overline{p}_2)} = CR_{12}$$

which can be written as

$$g(k) = F(k) - M(k) = CR_{12} \tag{40}$$

We note that $g(k)$ is a strictly increasing function of k for all $k > 0$, since its first-order derivative is

$$g'(k) = \frac{1}{k^2} \int_0^k x f(x) dx > 0$$

Therefore, if the solution to (40) exists, it is unique.

Using $\frac{\partial J}{\partial v_1} = 0$, $F(k) - M(k) = CR_{12}$ and $v_1 + v_2 = k$, we obtain v_2 as a function of k:

$$\begin{aligned} v_2 &= \frac{(\overline{h}_1 + \overline{p}_1) F(k) - \overline{p}_1}{\frac{\Delta}{k} M(k)} \\ &= \left(\frac{CR_2 - CR_{12}}{CR_2 - CR_1} \right) \left(\frac{F(k) - CR_1}{F(k) - CR_{12}} \right) k \end{aligned}$$

Similarly, using $\frac{\partial J}{\partial v_2} = 0$, $F(k) - M(k) = CR_{12}$ and $v_1 + v_2 = k$, we obtain

$$v_1 = \left(\frac{CR_1 - CR_{12}}{CR_2 - CR_1}\right)\left(\frac{CR_2 - F(k)}{F(k) - CR_{12}}\right) k$$

To ensure $v_1 > 0$ and $v_2 > 0$, $F(k)$ needs to satisfy certain conditions. In particular, if $\overline{h}_1/\overline{h}_2 > \overline{p}_1/\overline{p}_2$, then $CR_1 < F(k) < CR_2$; if $\overline{h}_1/\overline{h}_2 < \overline{p}_1/\overline{p}_2$, then $CR_2 < F(k) < CR_1$, by equation (11).

Proof of Lemma 2

In the complementarity conditions (35) and (36), let us suppose $v_1 > 0$ and $v_2 = 0$. Then, we must have $\frac{\partial J}{\partial v_1} = 0$ and $\lambda_2 = -\frac{\partial J}{\partial v_2} \geq 0$. That is,

$$\frac{\partial J}{\partial v_1} = \overline{p}_1 - (\overline{h}_1 + \overline{p}_1)F(v_1) = 0$$

Therefore we obtain $F(v_1) = CR_1$. Also we have

$$\frac{\partial J}{\partial v_2} = \overline{p}_2 - (\overline{h}_2 + \overline{p}_2)F(v_1) - \Delta M(v_1) \leq 0$$

Dividing the both sides by $(\overline{h}_2 + \overline{p}_2)$, we obtain

$$CR_2 - CR_1 - \frac{\Delta}{\overline{h}_2 + \overline{p}_2} M(v_1) \leq 0$$

or

$$M(v_1) \geq (CR_2 - CR_1)\frac{\overline{h}_2 + \overline{p}_2}{\Delta} = CR_1 - CR_{12}$$

Proof of Lemma 3

In the complementarity conditions (35) and (36), let us suppose $v_1 = 0$ and $v_2 > 0$. Then, we must have $\lambda_1 = -\frac{\partial J}{\partial v_1} \geq 0$ and $\frac{\partial J}{\partial v_2} = 0$. From these two conditions, we obtain the following results as similar to the proof of Lemma 2:

$$F(v_2) = CR_2$$
$$M(v_2) \leq CR_2 - CR_{12}$$

Proof of Proposition 3

When $v_1 > 0$ and $v_2 > 0$ satisfy the KKT conditions (Case 1), then $k = v_1 + v_2$ is uniquely determined. Therefore, (v_1, v_2) is also uniquely determined. Let us now consider a point $(u_1, u_2) = (v_1 + \delta, v_2 - \delta)$ for a constant δ. We note $u_1 + u_2 = k$. Therefore we have $\mathbb{E}[R(v_1, v_2)] = \mathbb{E}[R(u_1, u_2)]$.

However, we know (u_1, u_2) is not a local optimum (it does not satisfy the KKT conditions). That is, there exists a point $(w_1, w_2) \in \mathcal{N}_{\varepsilon_1}(v_1, v_2)$ such that $\mathbb{E}[R(w_1, w_2)] > \mathbb{E}[R(u_1, u_2)]$ for all $\varepsilon_1 > 0$, where $\mathcal{N}_\varepsilon$ denotes the ε-neighborhood. By making $|\delta|$ and ε_1 arbitrarily small, we can make $(w_1, w_2) \in \mathcal{N}_{\varepsilon_2}(v_1, v_2)$ for any $\varepsilon_2 > 0$ and $\mathbb{E}[R(w_1, w_2)] > \mathbb{E}[R(v_1, v_2)]$. Therefore (v_1, v_2) is not a local optimum.

Calculation of $\phi(v)$ in equation (23) in Section 4.1

$R(v) - R(v^*) = p(v - v^*) - h\{\max(v - X, 0) - \max(v^* - X, 0)\} + q\{\max(X - v, 0) - \max(X - v^*, 0)\}$

Depending on the range of X, the value of right hand side of equation (22) is:

$$R(v) - R(v^*) = \begin{cases} \overline{h}(v^* - v) & \text{if } X \leq \min(v^*, v) \\ (\overline{h} + \overline{p})X - \overline{h}v - \overline{p}v^* & \text{if } v^* < X \leq v \\ -X(\overline{h} + \overline{p}) + \overline{p}v + \overline{h}v^* & \text{if } v < X \leq v^* \\ \overline{p}(v - v^*) & \text{if } \max(v^*, v) < X \end{cases}$$

where $\overline{p} = p - q$ and $\overline{h} = h - p$. Suppose, $v > v^*$, then we have

$$R(v) - R(v^*) = \begin{cases} \overline{h}(v^* - v) & \text{if } X \leq v^* \\ (\overline{h} + \overline{p})X - \overline{h}v - \overline{p}v^* & \text{if } v^* < X \leq v \\ \overline{p}(v - v^*) & \text{if } v < X \end{cases}$$

Therefore

$$R(v) - R(v^*) \leq 0 \quad \text{if} \quad X < \frac{\overline{h}v + \overline{p}v^*}{\overline{h} + \overline{p}}$$
$$R(v) - R(v^*) > 0 \quad \text{if} \quad X \geq \frac{\overline{h}v + \overline{p}v^*}{\overline{h} + \overline{p}}$$

Consequently, we obtain

$$\phi(v) = \Pr\left[X \leq \frac{\overline{h}v + \overline{p}v^*}{\overline{h} + \overline{p}}\right] = F\left(\frac{\overline{h}v + \overline{p}v^*}{\overline{h} + \overline{p}}\right) \quad \text{if} \quad v > v^*$$

With similar analysis, the exact expression for equation (21) becomes:

$$\phi(v) = \begin{cases} F\left(\frac{\overline{h}v + \overline{p}v^*}{\overline{h} + \overline{p}}\right) & \text{if } v > v^* \\ 1 - F\left(\frac{\overline{h}v^* + \overline{p}v}{\overline{h} + \overline{p}}\right) & \text{if } v < v^* \\ 1 & \text{if } v = v^* \end{cases}$$

Calculation of $\phi(v_1, v_2)$ in equations (26) and (27) in Section 4.2

The values of ϕ can be obtained by considering $R(v_1, v_2) - R({v_1}^*, {v_2}^*)$ with the range of X:

$$R(v_1,v_2)-R({v_1}^*,{v_2}^*) = \begin{cases} -\left[\overline{h}_1(v_1-\chi_1{v_1}^*)+\overline{h}_2(v_2-\chi_2{v_2}^*)\right] \\ +\frac{h_1v_1+h_2v_2-(\chi_1h_1+\chi_2h_2)(v_1+v_2)}{v_1+v_2}X & \text{if } X \le \min[(v_1+v_2), \chi_1{v_1}^*+\chi_2{v_2}^*] \\ \\ -(\overline{h}_1v_1+\overline{h}_2v_2+\chi_1\overline{p}_1{v_1}^*+\chi_2\overline{p}_2{v_2}^*) \\ +\frac{(\overline{h}_1+\overline{p}_1)v_1+(\overline{h}_2+\overline{p}_2)v_2}{v_1+v_2}X & \text{if } \chi_1{v_1}^*+\chi_2{v_2}^* < X \le (v_1+v_2) \\ \\ \overline{p}_1v_1+\overline{p}_2v_2+\chi_1\overline{h}_1{v_1}^*+\chi_2\overline{h}_2{v_2}^* \\ -(\chi_1h_1+\chi_2h_2-q)X & \text{if } (v_1+v_2) < X \le \chi_1{v_1}^*+\chi_2{v_2}^* \\ \\ \overline{p}_1v_1+\overline{p}_2v_2-(\chi_1\overline{p}_1{v_1}^*+\chi_2\overline{p}_2{v_2}^*) & \text{if } \max[(v_1+v_2), \chi_1{v_1}^*+\chi_2{v_2}^*] < X \end{cases}$$

Let us define

$$A = \frac{(v_1+v_2)\left[\overline{h}_1(v_1-\chi_1{v_1}^*)+\overline{h}_2(v_2-\chi_2{v_2}^*)\right]}{h_1v_1+h_2v_2-(\chi_1h_1+\chi_2h_2)(v_1+v_2)}$$
$$B = \frac{(v_1+v_2)(\overline{h}_1v_1+\overline{h}_2v_2+\chi_1\overline{p}_1{v_1}^*+\chi_2\overline{p}_2{v_2}^*)}{(\overline{h}_1+\overline{p}_1)v_1+(\overline{h}_2+\overline{p}_2)v_2}$$
$$C = \frac{\overline{p}_1v_1+\overline{p}_2v_2+\chi_1\overline{h}_1{v_1}^*+\chi_2\overline{h}_2{v_2}^*}{\chi_1h_1+\chi_2h_2-q}$$

We can derive ϕ by considering two cases: (1) $v_1+v_2 \ge \chi_1{v_1}^*+\chi_2{v_2}^*$ and (2) $v_1+v_2 < \chi_1{v_1}^*+\chi_2{v_2}^*$. Consider the case $v_1+v_2 \ge \chi_1{v_1}^*+\chi_2{v_2}^*$. We can obtain the result that $R(v_1, v_2) \le R({v_1}^*, {v_2}^*)$ only when one of the following set of conditions is met:

$$\begin{array}{rlll} \text{(i)} & X \in [0, \chi_1{v_1}^*+\chi_2{v_2}^*] & \text{and} & X \in [0, A] \\ \text{(ii)} & X \in (\chi_1{v_1}^*+\chi_2{v_2}^*, v_1+v_2] & \text{and} & X \in [\chi_1{v_1}^*+\chi_2{v_2}^*, B] \\ \text{(iii)} & X \in (v_1+v_2, \infty) & \text{and} & \overline{p}_1v_1+\overline{p}_2v_2 \le \chi_1\overline{p}_1{v_1}^*+\chi_2\overline{p}_2{v_2}^* \end{array}$$

Similarly, for the case $v_1+v_2 < \chi_1{v_1}^*+\chi_2{v_2}^*$ one of the following conditions needs to be satisfied:

$$\begin{array}{rlll} \text{(i)} & X \in [0, v_1+v_2] & \text{and} & X \in [0, A] \\ \text{(ii)} & X \in (v_1+v_2, \chi_1{v_1}^*+\chi_2{v_2}^*] & \text{and} & X \in [C, \infty) \\ \text{(iii)} & X \in (\chi_1{v_1}^*+\chi_2{v_2}^*, \infty) & \text{and} & \overline{p}_1v_1+\overline{p}_2v_2 \le \chi_1\overline{p}_1{v_1}^*+\chi_2\overline{p}_2{v_2}^* \end{array}$$

From the above results, for $\overline{p}_1v_1+\overline{p}_2v_2 \le \chi_1\overline{p}_1{v_1}^*+\chi_2\overline{p}_2{v_2}^*$, we can obtain:

$$\phi(v_1,v_2) = \begin{cases} F[\max\{\min(A, \chi_1{v_1}^*+\chi_2{v_2}^*), 0\}]+1-F[v_1+v_2] \\ +F[\max\{\min(v_1+v_2, B), \chi_1{v_1}^*+\chi_2{v_2}^*\}]-F[\chi_1{v_1}^*+\chi_2{v_2}^*] & \text{if } v_1+v_2 \ge \chi_1{v_1}^*+\chi_2{v_2}^* \\ \\ F[\max\{\min(A, v_1+v_2), 0\}]+1 \\ -F[\max\{\min(\chi_1{v_1}^*+\chi_2{v_2}^*, C), v_1+v_1\}] & \text{if } v_1+v_2 < \chi_1{v_1}^*+\chi_2{v_2}^* \end{cases}$$

Otherwise,

$$\phi(v_1, v_2) = \begin{cases} F[\max\{\min(A, \chi_1 {v_1}^* + \chi_2 {v_2}^*), 0\}] \\ +F[\max\{\min(v_1+v_2, B), \chi_1 {v_1}^* + \chi_2 {v_2}^*\}] - F[\chi_1 {v_1}^* + \chi_2 {v_2}^*] & \text{if } v_1+v_2 \geq \chi_1 {v_1}^* + \chi_2 {v_2}^* \\ \\ F[\max\{\min(A, v_1+v_2), 0\}] \\ +F[\chi_1 {v_1}^* + \chi_2 {v_2}^*] - F[\max\{\min(\chi_1 {v_1}^* + \chi_2 {v_2}^*, C), v_1+v_1\}] & \text{if } v_1+v_2 < \chi_1 {v_1}^* + \chi_2 {v_2}^* \end{cases}$$

Proof of Lemma 4

Consider v_2 is constant. Differentiating $\mathbb{E}[R(v_1, v_2)]$ with respect to v_1, we obtain:

$$\frac{\partial \mathbb{E}[R(v_1, v_2)]}{\partial v_1} = \overline{p}_1 - (\overline{h}_1 + \overline{p}_1)F(v_1+v_2) + \frac{\{(\overline{h}_1 + \overline{p}_1) - (\overline{h}_2 + \overline{p}_2)\}v_2}{(v_1+v_2)^2} \int_0^{v_1+v_2} xf(x)dx$$

Differentiating again, we get

$$\frac{\partial^2 \mathbb{E}[R(v_1, v_2)]}{\partial {v_1}^2} = -\left[\frac{f(v_1+v_2)}{v_1+v_2}[(\overline{h}_1 + \overline{p}_1)v_1 + (\overline{h}_2 + \overline{p}_2)v_2] + \frac{2(h_1-h_2)v_2}{(v_1+v_2)^2} \int_0^{v_1+v_2} xf(x)dx\right] < 0$$

Similarly, when v_1 is constant, we can prove that $\frac{\partial^2 \mathbb{E}[R(v_1, v_2)]}{\partial {v_2}^2} \leq 0$.

Proof of Proposition 4

Suppose v_1 is constant. Since $\mathbb{E}[R(v_1, v_2)]$ is concave with respect to v_2 by Lemma 4, we have

$$\left.\frac{\partial \mathbb{E}[R(v_1, v_2)]}{\partial v_2}\right|_{\bar{v}_2 > \tilde{v}_2} < 0$$

Therefore, $\mathbb{E}[R(v_1, v_2)]$ is decreasing for $v_2 > \tilde{v}_2$.

For case 1 and 2 of Table 1, the neutral solution is $\left(F^{-1}\left\{\frac{p_1-q}{h_1-q}\right\}, 0\right)$. Therefore,

$$\begin{aligned} \mathbb{E}\left[R\left(F^{-1}\left\{\frac{p_1-q}{h_1-q}\right\}, 0\right)\right] &= (p_1-q)v_1 + q\mu - (h_1-q)v_1\frac{p_1-q}{h_1-q} + (h_1-q)\int_0^{F^{-1}\left\{\frac{p_1-q}{h_1-q}\right\}} xf(x)dx \\ &= q\mu + (h_1-q)\int_0^{F^{-1}\left\{\frac{p_1-q}{h_1-q}\right\}} xf(x)dx > R_0 \end{aligned}$$

Similarly, for case 3 of Table 1, we have

$$\mathbb{E}\left[R\left(0, F^{-1}\left\{\frac{p_2-q}{h_2-q}\right\}\right)\right] = q\mu + (h_2-q)\int_0^{F^{-1}\left\{\frac{p_2-q}{h_2-q}\right\}} xf(x)dx > R_0$$

In both cases, we have $\mathbb{E}[R(v_1, \bar{v}_2)] < R_0 < \mathbb{E}[R(v_1^*, v_2^*)]$ and Constraint of the problem (28) is never satisfied. Therefore, for each v_1 fixed, any v_2 such that $v_2 > \tilde{v}_2(v_1)$ and $\mathbb{E}[R(v_1, \tilde{v}_2(v_1))] < R_0$ is not a solution to problem (28). Similar result holds vice versa. Now, we have:

$$\mathbb{E}[R(0,\bar{v}_2)] = q\mu + \left[(p_2-q)\bar{v}_2 - (h_2-q)\bar{v}_2 F[\bar{v}_2] + (h_2-q)\int_0^{F^{-1}[\bar{v}_2]} xf(x)dx\right]$$
$$= R_0 + \left[(p_2-q)\bar{v}_2 - (h_2-q)\bar{v}_2 F[\bar{v}_2] + (h_2-q)\int_0^{F^{-1}[\bar{v}_2]} xf(x)dx\right] \leq R_0$$

Therefore, $(p_2-q)\bar{v}_2 - (h_2-q)\bar{v}_2 F[\bar{v}_2] + (h_2-q)\int_0^{F^{-1}[\bar{v}_2]} xf(x)dx \leq 0$. As $\mathbb{E}[R(0,\bar{v}_2)]$ is decreasing for $v_2 \geq \bar{v}_2$, any increase in total contract size will further decrease the $\mathbb{E}[R]$, i.e. $\mathbb{E}[R(v_1,\bar{v}_2)] < \mathbb{E}[R(0,\bar{v}_2)]$.