Evolutionary and Preferential Attachment Models of Demand Growth and Their Use in Transportation Planning

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Abstract

This paper considers a dual time scale transportation planning model for which demand evolves on a day-to-day time scale and traffic flows on arcs of the transportation network fluctuate on a within-day time scale. We show the problem is particularly amenable to a demand learning approach when using a demand model motivated by evolutionary game theory. We then argue that an alternative model of demand growth based on the paradigm of preferential attachment familiar from network science may be fit into the same mathematical structure and computed with equal efficiency. Numerical experiments for comparing the impact of these two models of day-to-day demand dynamics are proposed.

1 Introduction

It is widely acknowledged that, to create models for transportation planning that recognize the essential dynamic character of passenger network flows, one must consider two time scales: the so-called within-day time scale and the day-to-day time scale. Substantial progress has been made in modeling within-day dynamic flows for fixed trip matrices; one of the most widely acknowledged models for this purpose is the dynamic user equilibrium model proposed by Friesz et al. (1993) and studied by Xu et al. (1999), Wu et al. (1998), Friesz et al. (2001), Bliemer and Bovy (2003), and Friesz and Mookherjee (2006). In this paper we propose two day-to-day models of demand growth compatible with a differential variational inequality formulation of the Friesz et al. (1993) model. The first of these employs dynamics inspired by evolutionary game theory, while the second uses the perspective of preferential attachment familiar from the network science

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and social network literature to create a model of demand growth. Additionally, numerical experiments to compare and contrast the two proposed theories of demand growth are described, along with hypotheses that one might address via such experiments.

2 Dynamic User Equilibrium

First, however, we need to make a few comments about modeling and computing dynamic flow patterns on traffic networks. *Dynamic traffic assignment* is the name given to the determination of time varying traffic flows for road networks. When those flows obey a differential Nash-like equilibrium relative to departure rates and route choice, we say we have a dynamic user equilibrium flow pattern. To define a dynamic user equilibrium, we introduce the notion of an effective path delay operator $\Psi_p(t, h)$, which expresses the unit path delay for departure time t and traffic conditions h. The vector h is time dependent and its p^{th} component is $h_p(t)$, the departure rate from the origin of path p at time t. A dynamic user equilibrium flow pattern has the property that

$$h_p^* > 0, p \in P_{ij} \Longrightarrow \Psi_p(t, h^*) = v_{ij} \tag{1}$$

where P_{ij} is the set of paths that connect origin-destination pair $(i, j) \in W$, while W is the set of all origin-destination pairs. Furthermore, v_{ij} is the minimum travel delay that can be experienced for $(i, j) \in W$. Embedded within each effective path delay operator $\Psi_p(t, h)$ is a notion of arc delay (congestion) for the arcs comprising a given path and a penalty for early/late arrival. In fact the path delay operators are really a shorthand for a separate model, frequently called a network loading model, which determines the propagation of flows through a given network, as well as the path delays experienced, in response to a given vector of departure rates.

Additionally all path-specific departure rates are non-negative so we write

$$h = (h_p : p \in P) \geq 0 \tag{2}$$

where P is the set of all network paths. As a consequence

$$\Psi_p(t, h^*) > v_{ij}, p \in P_{ij} \Longrightarrow h_p^* = 0$$

as can easily be proven from (1) by contradiction. We next comment that the relevant notion of flow conservation is

$$\sum_{p \in P_{ij}} \int_0^T h_p(t)\, dt = Q_{ij} \quad \forall (i, j) \in W$$

where Q_{ij} is the fixed travel demand (expressed as a traffic volume) for $(i, j) \in W$. Thus, the set of feasible solutions is

$$\Lambda = \left\{ h > 0 : \sum_{p \in P_{ij}} \int_0^T h_p(t)\, dt = Q_{ij} \quad \forall (i, j) \in W \right\} \tag{3}$$

Friesz et al. (1993) show that a dynamic user equilibrium is equivalent to the following variational inequality:

$$\left.\begin{array}{c} \text{find } h^* \in \Lambda \text{ such that} \\ \sum_{p\in P} \displaystyle\int_0^T \Psi_p(t,h^*) \left(h_p - h_p^*\right) dt \geq 0 \quad \forall h \in \Lambda \end{array}\right\} \tag{4}$$

Suffice it to say that algorithms exist for solving (4); these include the fixed point algorithm presented and tested in this paper.

3 Demand Dynamics

Consider a transportation network for which a set W of origin-destination pairs (i,j) have been defined. Let $\tau \in \Upsilon \equiv \{1,2,...,L\}$ be one typical discrete day relative, and take the length of each day to be Δ, while the continuous clock time t within each day is $t \in [(\tau-1)\Delta, \tau\Delta]$ for all $\tau \in \{1,2,...,L\}$. The entire planning horizon spans L consecutive days. As noted above, we assume the travel demand for each day changes based on the moving average of congestion experienced over previous days.

3.1 The Dual Time Scale Model

Let us suppose we have a demand growth model of the abstract form

$$\begin{aligned} Q_{ij}^{\tau+1} &= \mathcal{F}_{ij}\left(Q_{ij}^{\tau}, h^{\tau}, \theta\right) \quad \forall\,(i,j) \in W,\ \tau \in \{0,1,2,...L-1\} \\ Q_{ij}^{\tau} &\geq 0 \quad \forall\,(i,j) \in W,\ \tau \in \{0,1,2,...L\} \\ Q_{ij}^{0} &= K_{ij}^{0} \quad \forall\,(i,j) \in W \end{aligned}$$

where

$$\begin{aligned} Q_{ij}^{\tau} &= \text{origin-destination travel demand } \forall\,(i,j) \in W,\ \tau \in \{0,1,2,...L\} \\ Q^{\tau} &= \left(Q_{ij}^{\tau} : (i,j) \in W\right) \\ Q &= (Q^{\tau} : \tau \in \Upsilon) \\ K_{ij}^{0} &= \text{a known, non-negative constant } \forall\,(i,j) \in W \\ h^{\tau} &= \left(h_p^{\tau} : p \in P\right) \\ h &= (h^{\tau} : \tau \in \Upsilon) \\ \theta &= \text{a vector of model parameters} \end{aligned}$$

Also we define

$$\begin{aligned} \Lambda_\tau(Q^\tau) &= \left\{ h^\tau > 0 : \sum_{p\in P_{ij}} \int_0^T h_p^\tau(t)\,dt = Q_{ij}^\tau \quad \forall\,(i,j) \in W \right\} \\ \Lambda(Q) &= \prod_{\tau=1}^{L} \Lambda_\tau(Q^\tau) \end{aligned}$$

Then a dual time scale model of dynamic user equilibrium with endogenous demand growth is

$$\left.\begin{array}{c}\text{find } Q \geq 0 \text{ and } h^* \in \Lambda\left(Q\right) \text{ such that} \\ \sum_{p\in P} \int_0^T \Psi_p(t, h^{\tau *}) \left(h_p^\tau - h_p^{\tau *}\right) dt \geq 0 \quad \forall \tau \in \Upsilon, h^\tau \in \Lambda_\tau\left(Q^\tau\right) \\ Q_{ij}^{\tau+1} = \mathcal{F}_{ij}\left(Q_{ij}^\tau, h, \theta\right) \qquad \forall\left(i,j\right) \in W \\ Q_{ij}^0 = K_{ij}^0 \qquad \forall\left(i,j\right) \in W \end{array}\right\} \tag{5}$$

This model may be solved by time stepping, so that exactly one variational inequality is faced for each value of τ.

3.2 An Ad Hoc Model of Demand Growth

We postulate that the travel demands Q_{ij}^τ for day τ between a given OD pair $(i,j) \in W$ are determined by the following system of difference equations:

$$Q_{ij}^{\tau+1} = \left[Q_{ij}^\tau - s_{ij}^\tau \left\{ \frac{\displaystyle\sum_{p\in P_{ij}} \sum_{j=0}^{\tau-1} \int_{j\cdot\Delta}^{(j+1)\cdot\Delta} \Psi_p\left[t, x\left(h^*, g^*\right)\right] dt}{|P_{ij}| \cdot \tau \cdot \Delta} - \chi_{ij} \right\} \right]^+$$

$$\forall \tau \in \{0, 1, 2, ...L-1\} \tag{6}$$

with boundary condition

$$Q_{ij}^0 = \tilde{Q}_{ij} \tag{7}$$

where $\tilde{Q}_{ij} \in \Re_+^1$ is the fixed travel demand for the OD pair $(i,j) \in W$ for the first day and χ_{ij} is the so-called fitness level. The operator $[x]^+$ is shorthand from $\max[0, x]$. The parameter s_{ij}^τ is related to the rate of change of inter-day travel demand. The above system of difference equations assumes that the moving average of effective travel delay plus any imposed toll is the principal signal that influences demand learning.

3.3 Replicator Dynamics for Demand Growth

The model (6) does not precisely capture the structure proposed by Hofbauer and Sigmund (1998) for the fundamental dynamics of evolutionary game theory, namely replicator dynamics. For a state variable Q, replicator dynamics have the structural form

$$\frac{\dot{Q}}{Q} = \alpha\left\{\text{fitness} - \text{average fitness}\right\} \tag{8}$$

where α is a constant of proportionality and the notion of fitness of a given system is given a broad interpretation. We may modify the story behind (6) to more closely correspond with (8) by writing

$$\frac{Q_{ij}^{\tau+1}-Q_{ij}^{\tau}}{Q_{ij}^{\tau}} = \alpha_{ij}^{\tau}\left\{\chi_{ij}-\frac{\sum\limits_{p\in P_{ij}}\sum\limits_{j=0}^{\tau-1}\int_{j\cdot\Delta}^{(j+1)\cdot\Delta}\Psi_p\left[t,h^*(t)\right]dt}{|P_{ij}|\cdot\tau\cdot\Delta}\right\}$$

$$\forall\tau \in \{0,1,2,...L-1\} \tag{9}$$

with the same boundary condition (7). We now introduce a specific definition of instantaneous fitness. In particular, we assume instantaneous fitness for a given origin-destination pair $(i,j)\in W$ is

$$\chi_{ij}\equiv v_{ij}=\min_{p\in P_{ij}}\Psi_p\left[\tau,h^*(\tau)\right] \tag{10}$$

In words, instantaneous fitness is least travel delay achieved at the end of the previous discrete time period (yesterday) and hence known at the start of the current discrete time period (today). Obviously (9) may be manipulated to give

$$Q_{ij}^{\tau+1} = Q_{ij}^{\tau}-\alpha_{ij}^{\tau}\left\{\frac{\sum\limits_{p\in P_{ij}}\sum\limits_{j=0}^{\tau-1}\int_{j\cdot\Delta}^{(j+1)\cdot\Delta}\Psi_p\left[t,h^*(t)\right]dt}{|P_{ij}|\cdot\tau\cdot\Delta}-v_{ij}\right\}Q_{ij}^{\tau}$$

$$\forall\tau \in \{0,1,2,...L-1\} \tag{11}$$

If information technology is increasing the speed of access to data about least travel delay, then the length of each "day" can be shortened, as the notion of day used herein is arbitrary. If one wishes to assure demand does not become negative, then (11) is replaced by

$$Q_{ij}^{\tau+1} = \left[Q_{ij}^{\tau}-\alpha_{ij}^{\tau}\left\{\frac{\sum\limits_{p\in P_{ij}}\sum\limits_{j=0}^{\tau-1}\int_{j\cdot\Delta}^{(j+1)\cdot\Delta}\Psi_p\left[t,h^*(t)\right]dt}{|P_{ij}|\cdot\tau\cdot\Delta}-v_{ij}\right\}Q_{ij}^{\tau}\right]^{+}$$

$$\forall\tau \in \{0,1,2,...L-1\} \tag{12}$$

4 Demand Dynamics Based on Preferential Attachment

In network science affinity networks are widely thought to evolve according to the notion of preferential attachment. Bianconi and Barabasi (2001) suggest an improved form of preferential attachment they call *quenched noise*. In that model they denote the connectivity of node i by $k_i(t)$ and postulate an associated fitness parameter η_i that accounts for differences among nodes with regard to their potential to attract and sustain attachments. They view network growth as a process whereby a new node with its distinct fitness is added randomly to a network during each period of time considered. The probability that a new node will connect to node i already present in the network is taken to be

$$\pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j} \tag{13}$$

Accordingly node i will increase its connectivity at the rate

$$\frac{\partial k_i}{\partial t} = m \frac{\eta_i k_i}{\sum_j \eta_j k_j} \tag{14}$$

where m is the number of new arcs added upon introduction of a new node. An initial condition must be associated with each equation (14) in order for the system of partial differential equations created in this fashion to be numerically solved.

Our interest in the above version of the preferential attachment model lies in the fact that it suggests a relationship between an underlying social network and the formation of travel demand. In particular one may partition, without loss of generality, the nodes of an affinity network into spatially related subsets of nodes that correspond to origins or destinations; when a given social network node is both an origin and a destination, a copy of it can be made and included as a member of both categories. Thus, arcs added to the social network, in light of the partition just described, join origin-destination pairs. As each arc of the affinity network represents a "travel relationship", it also represents an increment to the corresponding origin-destination travel demand. In this way, the Bianconi-Barabàsi network growth model, when applied to a social network, becomes a model of travel demand growth.

The above observations not withstanding, it is not really possible to directly employ the mathematical analysis surrounding the quenched noise model within a dynamic traffic assignment or congestion pricing model because the Bianconi-Barabàsi model lacks the spatial and agent detail needed for transportation network modeling. As a consequence, we need to provide a separate articulation of travel demand induced by affinity network growth, based on preferential attachment, that involves the variables and concepts introduced in previous sections and includes randomness. To that end we propose the following model:

$$Q_{ij}^{\tau+1} = Q_{ij}^{\tau} + s_{ij}^{\tau} \frac{\eta_{ij} Q_{ij}^{\tau}}{\sum_{k \in \mathcal{N}_o} \eta_{kj} Q_{kj}^{\tau}} \qquad \forall \tau \in \{0, 1, 2, ...L-1\} \tag{15}$$

where all notation is as before but now fitness is doubly subscripted and appears as η_{ij} for each $(i,j) \in W$ and we use W_o to denote nodes that are origins. Importantly each s_{ij}^{τ} remains a random variable that is naturally suited for treatment by a learning process. Note also that (15) is a discrete time version of preferential attachment, in that demand growth is greatest for origin-destination pairs with the largest current demand. Variations of (15) are easily constructed. For instance

$$Q_{ij}^{\tau+1} = Q_{ij}^{\tau} + s_{ij}^{\tau} \frac{\sum_{k \in \mathcal{N}_o} \eta_{kj} Q_{kj}^{\tau}}{\sum_{(k,\ell) \in W} \eta_{k\ell} Q_{k\ell}^{\tau}} \qquad \forall \tau \in \{0, 1, 2, ...L-1\} \tag{16}$$

considers all origin-destination pairs in assessing the probability of a new increment in demand for a given pair.

Note that both model (15) and model (16) have the property that demand grows monotonically, which cannot be deemed realistic for all time. Thus, a needed modification is the introduction of a term that corresponds to the retirement of individuals from the underlying social network; if the rate of such retirements is ρ_{ij}^{τ}, then (15) and (16) may be re-stated, respectively, as

$$Q_{ij}^{\tau+1} = \left[Q_{ij}^{\tau} + s_{ij}^{\tau} \frac{\eta_{ij} Q_{ij}^{\tau}}{\sum_{k \in \mathcal{N}_o} \eta_{kj} Q_{kj}^{\tau}} - \rho_{ij}^{\tau} Q_{ij}^{\tau} \right]^{+} \qquad \forall \tau \in \{0, 1, 2, ...L-1\} \tag{17}$$

$$Q_{ij}^{\tau+1} = \left[Q_{ij}^{\tau} + s_{ij}^{\tau} \frac{\sum_{k \in \mathcal{N}_o} \eta_{kj} Q_{kj}^{\tau}}{\sum_{(k,\ell) \in W} \eta_{k\ell} Q_{k\ell}^{\tau}} - \rho_{ij}^{\tau} Q_{ij}^{\tau} \right]^{+} \qquad \forall \tau \in \{0, 1, 2, ...L-1\} \tag{18}$$

where we have introduced the $[\cdot]^{+}$ operator to assure demand does not become negative. The retirement rates ρ_{ij}^{τ} may be determined empirically, by a separate model or randomly. The random approach seems more in keeping with notion of preferential attachment we have borrowed from network science to describe the addition of new demand.

5 Demand Learning via the Kalman Filter

In this section we will base our remarks on the *ad hoc* model (6). However, it should be clear that a completely analogous discussion of demand learning may be crafted for each of the demand models suggested above. The model parameters s_{ij}^{τ} will typically be unknown to the modeler and follow stochastic distributions. Assuming that the modelling error and observation error follow normal distributions, in this section, we adapt a well-known forecasting method, so-called Kalman filtering. Recall the *ad hoc* day-to-day dynamics for travel demand:

$$Q_{ij}^{\tau+1} = \left[Q_{ij}^{\tau} - s_{ij}^{\tau} \left\{ \frac{\sum_{p \in P_{ij}} \sum_{j=0}^{\tau-1} \int_{j \cdot \Delta}^{(j+1) \cdot \Delta} \Psi_p \left[t, x\left(h^*, g^*\right) \right] dt}{|P_{ij}| \cdot \tau \cdot \Delta} - \chi_{ij} \right\} \right]^{+}$$

$$\forall \tau \in \{0, 1, 2, ...L-1\} \tag{19}$$

Each parameter s_{ij}^{τ} is treated as fixed during the solution process, but it is stochastic and its real value is unknown. After one day is completed, we want to update the model parameter s_{ij}^{τ} to obtain a better estimate of demand for the next planning horizon. The dynamics of s_{ij}^{τ} are assumed to be

$$s_{ij}^{\tau+1} = s_{ij}^{\tau} + \xi_{ij}^{\tau}$$

where ξ_{ij}^{τ} is a random noise from a normal distribution $N(0, B_{ij})$. The matrix B_{ij} is known and called the process-noise covariance matrix.

The value of the parameter s_{ij}^{τ} cannot be observed directly but only through the change of realized travel demand, which can be defined as

$$z_{ij}^{\tau} \equiv \Delta Q_{ij}^{\tau} = Q_{ij}^{\tau+1} - Q_{ij}^{\tau}$$

Note that

$$\Delta Q_{ij}^{\tau} = -s_{ij}^{\tau} \left\{ \frac{\sum_{p \in P_{ij}} \sum_{j=0}^{\tau-1} \int_{j \cdot \Delta}^{(j+1)\cdot\Delta} \Psi_p \left[t, x\left(h^*, g^*\right)\right] dt}{|P_{ij}| \cdot \tau \cdot \Delta} - \chi_{ij} \right\} + \omega_{ij}^{\tau} \tag{20}$$

and ω_{ij}^{τ} is a random noise of observation from a normal distribution $N(0, R_{ij})$. The matrix R_{ij} is known and called the measurement noise covariance matrix. Referring to Section 12.6 Bryson and Ho (1975), we obtain the Kalman filter dynamics

$$\begin{aligned}
\bar{s}_{ij}^{\tau+1} &= \hat{s}_{ij}^{\tau} = \bar{s}_{ij}^{\tau} + V_{ij}^{\tau}\left[z_{ij}^{\tau} - H_{ij}^{\tau}\bar{s}_{ij}^{\tau}\right] \\
P_{ij}^{\tau} &= \left[\left(M_{ij}^{\tau}\right)^{-1} + \left(H_{ij}^{\tau}\right)^{T}\left(R_{ij}^{\tau}\right)^{-1} H_{ij}^{\tau}\right]^{-1} \\
M_{ij}^{\tau+1} &= P_{ij}^{\tau} + B_{ij}^{\tau}
\end{aligned}$$

where

$$\begin{aligned}
V_{ij}^{\tau} &\equiv P_{ij}^{\tau} H_{ij}^{\tau}\left(R_{ij}^{\tau}\right)^{-1} \\
H_{ij}^{\tau} &\equiv -\left\{ \frac{\sum_{p \in P_{ij}} \sum_{j=0}^{\tau-1} \int_{j \cdot \Delta}^{(j+1)\cdot\Delta} \Psi_p \left[t, x\left(h^*, g^*\right)\right] dt}{|P_{ij}| \cdot \tau \cdot \Delta} - \chi_{ij} \right\}
\end{aligned}$$

and $\bar{s}_{ij}^{\tau}$ is the *a priori* estimate of s_{ij}^{τ} (before observation) and $\hat{s}_{ij}^{\tau}$ is the *a posteriori* estimate (after observation). When estimation process based on the above dynamics is completed, we have $\bar{s}_{ij}^{\tau+1}$, which is the value of s_{ij} used in the next discrete time interval.

6 Conclusions

We have proposed some dynamics for the growth of travel demand in vehicular traffic networks. Clearly, these ideas are preliminary; they are meant to promote discussion and to motivate future research. A great deal of work still needs to be done.

6.1 Numerical Experiments

The models proposed above were constructed to conform with evolutionary game theory and the dynamics of preferential attachment in social networks. However, we do not know what spatial and temporal patterns of traffic flows and network usage at the link level will result from these models. In particular, we do not know if individual demand growth models, drawn from the family of models we have described, will display statistical tendencies to promote or diminish stability, resiliency, sustainability, congestion, the price of anarchy, the Braess paradox, and social cohesion.

For example if the rate of retirements in model (17) is described as a feedback mechanism driven by link-level congestion occurring on the transportation network, can preferential attachment dynamics maintain the level of connectivity of a community (origin) with other communities (destinations) sufficient to assure adequate employment and/or other means of sustainability? If the answer to this question were "no" based on numerous simulations, then empirical studies to ascertain whether demand does in fact grow by preferential attachment are needed. If such growth mechanisms are found to occur in the real world, then policies that deter demand growth by preferential attachment are warranted. Many other questions and hypotheses may be proposed and considered using the dual time scale model (5) together with one of the demand growth models of Sections 3.2, 3.3 and 4.

6.2 Other Network Growth Processes

The Bianconi-Barabàsi network growth model is only one of many that have been discussed in the network science literature. Several others are also worth considering in the current context. Erdos and Renyi (1959) start with a set of nodes and simply assume that each pair of nodes is connected by an arc with probability p. In the current setting, this corresponds to a situation in which the demand between an origin-destination pair increases by a fixed amount with probability p. One can complicate this model by specifying the number of destinations associated with each origin (i.e., by specifying the degree of each origin). The properties of the ensemble of graphs that have a given degree distribution have been studied by Molloy and Reed (1998), Newman et al. (2001), Chung and Lu (2004) and others. Finally, one might want to construct the network based on attributes of the network. For example, one might assume that realizations with lower cost are more likely to occur [as in the cost efficiency theory of Smith

(1983)] or one might make assumptions about the topological properties. These kinds of ensembles have been studied by Strauss (1986).

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