Valuation of American Options by the Gradient Projection Method

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Abstract

We study an equivalent optimization problem with an inequality constraint and boundary conditions, whose necessary condition for the optimality is the variational inequality presentation of American options. To solve the problem, we use the gradient projection method, with discretizations both in time and space. We tested the algorithm and compared with the projective successive over-relaxation method.

Key words: American options, variational inequalities, gradient projection

1 Introduction

An *American option* has a key feature that distinguishes it from a European option: exercise is permitted at any time during its life of the option. So, unlike a European option, we have to determine whether or not an American option should be exercised at each instant of time. Moreover, the valuation of an American option is a *free boundary problem*, which occurs in many engineering systems. This property was first pointed out by McKean [18].

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The valuation of American options with dividends has been studied by many scholars. Geske [10], Roll [20], and Whaley [21] obtained analytical solutions for the case of known discrete dividends, while Brennan and Schwartz [1] and Brennan and Schwartz [2] introduced the finite difference approximation approach with log-transformation. This numerical method approximates differential terms of the value function by discretizing both time and state space. The finite difference method is one of the most popular methods because it is flexible and easy to implement, so that non-standard forms of options also may be solved. Cox et al. [6] introduced the binomial method for the valuation of American options, which is also flexible and requires time discretizations. Geske and Shastri [12] summarized and compared these early methods.

Later, Geske and Johnson [11] presented an analytic solution to American put option with or without dividends. However, their formula is an infinite series that must be approximated by numerical methods. Kim [17] and Carr et al. [4] provided an integral representation of the option price. These methods are compared by Broadie and Detemple [3], who also derived the lower bound and upper bound for the value of American options.

More recent studies on American option pricing are based on linear complementarity problems (LCPs). Huang and Pang [14] provided discretized LCP formulations for various option problems including American options and suggested solution algorithms including projective successive over-relaxation (PSOR), Lemke's algorithm and a revised parametric principal pivoting (PPP) algorithm. Forsyth and Vetzal [9] considered a special penalty method for LCPs adequate to handle American option constraints, while Coleman et al. [5] proposed a Newton type method for a nonlinear programming problem based on quadratic penalization of the complementarity conditions. Ikonen and Toivanen [15] showed LU decomposition can improve the performance of several different algorithms for solving LCPs of American options.

Moreover, Dempster and Hutton [7] studied American option pricing problem using linear programming approach and Jaillet et al. [16] presented variational inequality formulation of American option pricing problem. In this paper, we will construct an extremal problem equivalent to the variational inequality formulation and discuss the gradient projection method for the extremal problem.

2 Linear Complementarity Problem Formulation of the American Options

It is well-known that an American put option pricing problem can be formulated as a linear complementarity problem: see Wilmott et al. [22] and Huang and Pang [14]. When we denote $P(s,t)$ the value of an American put option, we have a *linear complementarity problem* (LCP):

$$\mathcal{L}_{BS}(P(S,t)) \cdot [P(S,t) - \Phi(S)] = 0 \tag{1}$$
$$P(S,t) - \Phi(S) \geq 0 \tag{2}$$
$$-\mathcal{L}_{BS}(P(S,t)) \geq 0 \tag{3}$$

with boundary conditions

$$\begin{aligned} P(S,t) &\geq \Phi(S) \\ P(S,T) &= \Phi(S) \\ P(0,t) &= E \\ \lim_{S\to\infty} P(S,t) &= 0 \end{aligned}$$

where $\mathcal{L}_{BS}$ denote the Black-Scholes operator

$$\mathcal{L}_{BS} \equiv \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r-\delta)S\frac{\partial}{\partial S} - r$$

the pay-off function $\Phi(S)$ is defined by

$$\Phi(S) \equiv \max(E - S, 0)$$

Further, r is the interest rate, δ is the constant continuous dividend rate, σ is the volatility, S is the asset price, E is the strike price and T is the expiry date.

For American call options, similar LCP formulation is possible. Let $C(S,t)$ denote the value of an American call, then the LCP formulation is:

$$\begin{aligned} \mathcal{L}_{BS}(C(S,t)) \cdot [C(S,t) - \Phi(S)] &= 0 \\ C(S,t) - \Phi(S) &\geq 0 \\ -\mathcal{L}_{BS}(C(S,t)) &\geq 0 \end{aligned}$$

with boundary conditions

$$\begin{aligned} C(S,t) &\geq \Phi(S) \\ C(S,T) &= \Phi(S) \\ C(0,t) &= 0 \\ \lim_{S\to\infty} C(S,t) &\to \infty \end{aligned}$$

and in this case we have the pay-off function

$$\Phi(S) \equiv \max(S - E, 0)$$

3 Variational Inequality Formulation

In this section, we will formulate an American put option in a *variational inequality problem*. As an alternative approch to study mathematical programming problems, variational inequalities have been studied in various economic equilibrium problems. See Harker and Pang [13] and Facchinei and Pang [8] for general references of variational inequality problems and applications. In particular, Jaillet et al. [16] studied American option pricing problems in variational inequality form.

We define variational inequality problem as following:

Definition 1 *Given a nonempty set, Ω, and a function, $F : \Omega \to \mathbb{R}^n$, the variational inequality problem $VIP(F, \Omega)$ is to find a vector y such that*

$$\begin{aligned} y &\in \Omega \\ \langle F(y), x - y \rangle &\geq 0 \qquad \forall x \in \Omega \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the corresponding inner product.

Let us define a set of functions

$$\Omega = \{G(S,t) | G(S,t) - \Phi(S) \geq 0 \quad \forall S \in \mathbb{R}_+, t \in [0,T]\},$$

and pick $U \in \Omega$ so that

$$- \mathcal{L}_{BS}(P(S,t)) \cdot [U(S,t) - \Phi(S,t)] \geq 0 \qquad \forall S \in \mathbb{R}_+, t \in [0,T]. \quad (4)$$

We have also from (1)

$$-\mathcal{L}_{BS}(P(S,t)) \cdot [P(S,t) - \Phi(S)] = 0 \quad (5)$$

Subtraction (5) from (4), we get

$$- \mathcal{L}_{BS}(P(S,t)) \cdot [U(S,t) - P(S,t)] \geq 0 \qquad \forall S \in \mathbb{R}_+ \quad \forall t \in [0,T], \quad (6)$$

or, equivalently,

$$\int_0^\infty -\mathcal{L}_{BS}(P(S,t)) \cdot [U(S,t) - P(S,t)]\, dS \geq 0 \qquad \forall t \in [0,T], \quad (7)$$

which is a variational inequality formulation of an American put option. We note that the Ω is a nonempty square-integrable space where the corresponding norm is defined by

$$\langle u(S,t), v(S,t)\rangle \equiv \int_0^\infty [u(S,t)\cdot v(S,t)]\, dS$$

for any given instant of time $t \in [0, T]$.

3.1 Log Transformation

Let us consider the following transformation:

$$\begin{aligned} y &\equiv \log S \\ \tau &\equiv T - t \\ u(y,\tau) &\equiv P(S,t), \end{aligned}$$

Then (3) becomes

$$-\frac{\partial u}{\partial \tau} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial y^2} + \left(r - \delta - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial y} - ru \le 0. \tag{8}$$

Defining an operator

$$\Psi = \frac{\partial}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2} - \left(r - \delta - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial y} + r,$$

and the payoff function

$$\phi(y) = \max(E - e^y, 0),$$

we obtain a linear complementarity problem and variational inequality problem for this case; it is to find $u(y,\tau)$ such that, for all $v \in \Omega$

$$\Psi(u)\,[u - v] = 0, \Psi(u) \ge 0, u - v \ge 0$$

and the variational inequality problem is to find $u \in \Omega$ for each time instant $\tau \in [0, T]$ such that

$$\int_{-\infty}^{\infty} \Psi(u)\,[v - u]\, dy \ge 0 \quad \forall v \in \Omega, \tag{9}$$

where

$$\begin{aligned} \Omega = \{v : v - \phi \ge 0, v(y,0) = \phi(y),& \\ & v(-\infty,\tau) = \phi(-\infty), v(\infty,\tau) = \phi(\infty)\}. \end{aligned} \tag{10}$$

We denoted $f(\infty) = \lim_{x\to\infty} f(x)$.

4 An Extremal Problem in Continuous Time

Now we are interested in the articulation and direct solution of a functional mathematical program whose solutions are also solutions of (9). We show through numerical examples that such an approach is numerically efficient. Consider the extremal problem:

$$\min J(u) = \int_{-\infty}^{\infty} \int_{0}^{u} \Psi(v)\, dv dy \quad \text{s.t.} \quad u \in \Omega \tag{11}$$

where Ω is defined as (10). By deriving a necessary condition for this extremal problem, we recover variational inequality (9), thereby verifying that any solution of (11) is also a solution of (9). Therefore, any solution to (11), provided one exists, is a solution to the linear complementarity problem.

We will need some results of functional analysis to derive the necessary condition. First, we introduce the *Gateaux-differentiability.*

Definition 2 *Let V be a Hilbert space. A functional J is* Gateaux differentiable *or* G-differentiable *at $v \in V$ in the direction $\varphi \in V$, if the limit*

$$\lim_{\theta \to 0} \frac{J(v + \theta\varphi) - J(v)}{\theta}$$

exists. This limit is denoted by $\delta J(v, \varphi)$.

The famous Riesz's representation theorem leads to the introduction of the gradient. If J is G-differentiable at $v \in V$, and if $\delta J(v, \varphi)$ is a continuous linear form with respect to φ, then, there exists an element $\dfrac{\partial J}{\partial v} \in V$ such that

$$\forall \varphi \in V : \qquad \delta J(v, \varphi) = \left\langle \frac{\partial J}{\partial v}, \varphi \right\rangle .$$

Moreover, $\dfrac{\partial J}{\partial v}$ is called the *(Gateaux-) gradient of J at v.* See Section 10.3.3 of [19] for further discussion.

The validity of the extremal problem (11) may now be established. To this end, we must establish that the functional $J(u)$ is G-differentiable and the set Ω is convex. The relevant results are:

Lemma 1 *The functional*

$$J(u) = \int_{-\infty}^{\infty} \int_{0}^{u} \Psi(v) dv dy$$

is everywhere G-differentiable and

$$\frac{\partial J(u)}{\partial u} = \Psi(u)$$

Proof. We construct the G-derivative as

$$\begin{aligned}
\delta J(u,\varphi) &= \lim_{\theta\to 0}\frac{J(u+\theta\varphi)-J(u)}{\theta}\\
&= \int_{-\infty}^{\infty}\lim_{\theta\to 0}\frac{\int_0^{u+\theta\varphi}\Psi(v)dv-\int_0^{u}\Psi(v)dv}{\theta}dy\\
&= \int_{-\infty}^{\infty}\lim_{\theta\varphi\to 0}\frac{\int_0^{u+\theta\varphi}\Psi(v)dv-\int_0^{u}\Psi(v)dv}{\theta\varphi}\varphi dy\\
&= \int_{-\infty}^{\infty}\Psi(u)\varphi dy
\end{aligned}$$

Since

$$\delta J(u,\varphi) \doteq \left\langle \frac{\partial J}{\partial u},\varphi\right\rangle \doteq \int_{-\infty}^{\infty}\frac{\partial J}{\partial u}\varphi dy,$$

we have

$$\frac{\partial J}{\partial u} = \Psi(u)$$

■

Lemma 2 *The set Ω defined by (10) is convex.*

Proof. Pick $\bar{v}, \hat{v} \in \Omega$ so that

$$\bar{v} - \phi \geq 0, \qquad \hat{v} - \phi \geq 0$$

and define

$$v^{\lambda} = \lambda\bar{v} + (1-\lambda)\hat{v} \qquad \lambda \in [0,1].$$

Then $v^{\lambda} \in \Omega$. ■

Finally, we obtain the following theorem:

Theorem 1 *Any solution of the extremal problem (11) is a solution of the variational inequality (9).*

Proof. Let $v \in \Omega$ be arbitrary. Since Ω is convex, and $u \in \Omega$ implies

$$u + \theta(v-u) \in \Omega \qquad \forall\theta \in [0,1].$$

Hence for u to be a minimum of J on Ω it is necessary that $\forall v \in \Omega$

$$\left[\frac{d}{d\theta}J(u+\theta(v-u))\right]_{\theta=0} = \delta J(u, v-u) \geq 0.$$

Since J is G-differentiable at u and δJ is well-defined by Lemma 1, we have

$$\delta J(u, v-u) = \int_{-\infty}^{\infty} \Psi(u)(v-u)dy \geq 0 \qquad \forall v \in \Omega.$$

(9) follows immediately. ■

5 The Gradient Projection Algorithm

We study in this section the following projected gradient method:

Step 0. Initialization. *Set* $k = 0$. *Pick* $u^0(y, \tau) \in \Omega$.

Step 1. Determine gradient. *Calculate*

$$\begin{aligned} \frac{\partial J^k}{\partial u} \equiv \frac{\partial J\left(u^k\right)}{\partial u} &= \Psi(u^k) \\ &= \left[\frac{\partial u^k}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 u^k}{\partial y^2} \right. \\ &\quad \left. - \left(r - \delta - \frac{1}{2}\sigma^2\right)\frac{\partial u^k}{\partial y} + r u^k\right] \end{aligned}$$

Step 3. Update iterate. *Calculate*

$$u^{k+1} = P_{\Omega}\left\{u^k - \theta_k \frac{\partial J^k}{\partial u}\right\} = \max\left\{\phi, u^k - \theta_k \frac{\partial J^k}{\partial u}\right\}$$

where P_{Ω} *denotes the minimum norm projection onto* Ω *and* θ_k *is a variable scalar step.*

Step 4. Stopping test. *If an appropriate stopping test is satisfied, halt execution and declare*

$$u^*(y, \tau) \approx u^{k+1}(y, \tau)$$

Otherwise set $k = k + 1$ *and go to Step 1.*

For the convergence of this scheme and the detailed discussion, see the Chapter 10 in [19].

6 Finite Difference Approximation

In this section, we are interested in a finite approximation of infinite dimensional variational inequality problem (9). To recall

$$\int_{-\infty}^{\infty} \Psi(u)\,[v-u]\,dy \geq 0 \quad \forall v \in \Omega, \qquad \forall \tau \in [0,T] \tag{12}$$

We limit the domain of space y by an interval $[y_L, y_U]$ instead of $(-\infty, \infty)$ and discretize the interval by M sub intervals so that

$$\begin{aligned} y_i &= y_L + i\delta y, \qquad i = 0, ..., M \\ \delta y &= \frac{y_U - y_L}{M} \end{aligned}$$

Also we discretize the time by L intervals so that

$$\begin{aligned} \tau_j &= j\delta\tau, \qquad j = 0, ..., L \\ \delta\tau &= \frac{T}{L} \end{aligned}$$

Then (12) is approximated to

$$\sum_{i=0}^{M} \Psi(u_{i,j})\,[v_{i,j} - u_{i,j}] \geq 0 \quad \forall v \in \Omega, \qquad \forall j \in \{0, ..., L\} \tag{13}$$

where $u_{i,j} = u\,(y_i, \tau_j)$. By its nature, the finite difference approximation has an instability property which depends on the mesh sizes, δy and $\delta\tau$. When a finite difference approximation is unstable, the sequence of $u_{i,j}$'s is unbounded, hence the scheme fails to be convergent. This property is maily due to the accumulation of rounding errors. For the stability of the finite difference approximation discussed in this paper, we need $\delta\tau / (\delta y)^2$ bounded by a constant. See Wilmott et al. [22] for a further discussion.

Note that, for $j = 0$, that is $\tau = 0$, the VI (13) has the solution $u_{i,0} = \phi_i = \phi\,(y_i)$ from the initial condition. Starting from this solution for $j = 0$, we may solve (13) for the entire time domain step by step. Our next interest is, of course, how to approximate the parabolic operator $\Psi\,(\cdot)$. We may consider following approximations

$$
\begin{aligned}
\frac{\partial u_{i,j}}{\partial \tau} &\approx \frac{u_{i,j}-u_{i,j-1}}{\delta\tau} \\
\frac{\partial u_{i,j}}{\partial y} &\approx \theta\left(\frac{u_{i+1,j}-u_{i-1,j}}{2\delta y}\right) \\
&\quad + (1-\theta)\left(\frac{u_{i+1,j-1}-u_{i-1,j-1}}{2\delta y}\right) \\
\frac{\partial^2 u_{i,j}}{\partial y^2} &\approx \theta\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{(\delta y)^2}\right) \\
&\quad + (1-\theta)\left(\frac{u_{i+1,j-1}-2u_{i,j-1}+u_{i-1,j-1}}{(\delta y)^2}\right)
\end{aligned}
$$

where we used θ-approximation for the derivatives with respect to the space y. For $\theta = 0, \frac{1}{2}, 1$, the approximation becomes explicit, Crank-Nicolson, and implicit, respectively.

$$
\begin{aligned}
\Psi\left(u_{i,j}; u_{i,j-1}\right) &\approx \frac{u_{i,j}-u_{i,j-1}}{\delta\tau} \\
&- \frac{1}{2}\sigma^2\left[\theta\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{(\delta y)^2}\right)\right. \\
&\quad \left. + (1-\theta)\left(\frac{u_{i+1,j-1}-2u_{i,j-1}+u_{i-1,j-1}}{(\delta y)^2}\right)\right] \\
&- \left(r-\delta-\frac{1}{2}\sigma^2\right)\left[\theta\left(\frac{u_{i+1,j}-u_{i-1,j}}{2\delta y}\right)\right. \\
&\quad \left. + (1-\theta)\left(\frac{u_{i+1,j-1}-u_{i-1,j-1}}{2\delta y}\right)\right] \\
&+ r u_{i,j}
\end{aligned}
$$

where we denote $\Psi\left(u_{i,j}; u_{i,j-1}\right)$ the approximation of the operator $\Psi(\cdot)$ at $u_{i,j}$ given $u_{i,j-1}$ for all i and j.

As discussed above, the algorithm will be of the form:

(1) For $j = 0$, have the solution of the VI (13), that is, $u_{i,0} = \phi_i = \phi(y_i)$. Set $j = 1$.
(2) Given the solution $u_{i,j-1}$, solve the following VI

$$
\sum_{i=0}^{M} \Psi(u_{i,j}; u_{i,j-1})\left[v_{i,j}-u_{i,j}\right] \geq 0 \quad \forall v_{i,j} \in \Omega \tag{14}
$$

(3) Set $j = j+1$ and repeat Step 2 until $j = L$.

In this paper, the VIP (14) will be solved by the gradient projection method for the equivalent extremal problem. The gradient projection method is widely used for optimization problems and we found it is easy to impelement, in particular, for solving American option pricing problems when the method is combined with finite difference approximations.

7 Numerical Results

We have tested the gradient projection algorithm with discretizations both in time and space for several American call options. We used $y_L = -6.5$ and $y_U = 6.5$ for space discretization. Broadie and Detemple [3] studied upper and lower bounds for the values of American option, with which we compared our result in Table 1. The binomial method with 15,000 steps ([3]) is used to compare. To have accurate result, in the experiments represented in Table 1, very small mesh sizes were used. Also, the convergence property of the gradient projection algorithm was tested by experiments changing the mesh sizes. The result provided in Table 2 indeed shows the scheme converges as the number of meshes increases.

The performance of a popular method for the valuation of American options, the projected successive over-relaxation (PSOR) (see [22]), is compared with that of the gradient projection method. The computation result shown in Table 3 says that the gradient projection method is a competitive method in terms of accuracy and speed. Although a bigger step size in the gradient projection algorithm allows us to achieve a solution faster, we should take a smaller step size when the meshes are finer for the convergence. The over-relaxation parameters in the PSOR method and the step sizes in the gradient projection method are found by trial-and-error. For both methods, we stopped when the relative error based on the norm, $\left\| u^{k+1} - u^k \right\|$, is smaller than 10^{-5}. The computation times are averaged over 100 repeats.

The valuation of an American call option is presented graphically in Figures 1 and 2. All the computation in this paper was performed by MATLAB 7.0 at a generic desktop computer.

8 Conclusion

We have examined the gradient projection method for an equivalent extremal problem of the American option valuation. To this end, we first studied the linear complimentarity problem form for both American put and call options, and basic algebraic manipulations enabled us to have the variational inequality formulations. We used some results of functional analysis such as

Option Parameters	Asset Price	Lower Bound	Upper Bound	Binomial Method	Gradient Projection
$r = 0.03$	80.000	0.218	0.220	0.219	0.218
$\sigma = 0.20$	90.000	1.376	1.389	1.386	1.382
$\delta = 0.07$	100.000	4.750	4.792	4.783	4.777
	110.000	11.049	11.125	11.098	11.093
	120.000	20.000	20.061	20.000	20.001
$r = 0.03$	80.000	2.676	2.691	2.689	2.678
$\sigma = 0.40$	90.000	5.694	5.727	5.722	5.708
$\delta = 0.07$	100.000	10.190	10.250	10.239	10.223
	110.000	16.110	16.201	16.181	16.166
	120.000	23.271	23.392	23.360	23.347
$r = 0.00$	80.000	1.029	1.039	1.037	1.033
$\sigma = 0.30$	90.000	3.098	3.129	3.123	3.115
$\delta = 0.07$	100.000	6.985	7.051	7.035	7.026
	110.000	12.882	12.988	12.955	12.947
	120.000	20.650	20.779	20.717	20.713
$r = 0.07$	80.000	1.664	1.664	1.664	1.657
$\sigma = 0.30$	90.000	4.495	4.495	4.495	4.483
$\delta = 0.03$	100.000	9.251	9.251	9.251	9.237
	110.000	15.798	15.798	15.798	15.784
	120.000	23.706	23.706	23.706	23.695

Table 1
American call options with the expiry $T = 3$(year) and the strike price $E = 100$. We discretized in 400 intervals in time and 2000 intervals in space. (Step size $\theta_k = 0.0003$ is used.)

Asset Price	80	90	100	110
Number of Meshes (L, M)	Option Value			
$(50, 300)$	0.213	1.343	4.705	11.038
$(100, 500)$	0.214	1.361	4.741	11.067
$(150, 700)$	0.215	1.369	4.755	11.077
$(200, 1000)$	0.217	1.377	4.768	11.087
$(300, 1500)$	0.218	1.381	4.774	11.092
Binomial Method	0.219	1.386	4.783	11.098

Table 2
Values of an American call option with $T = 0.5$, $E = 100$, $r = 0.03$, $\sigma = 0.20$, $\delta = 0.07$ by different mesh sizes (Step size $\theta_k = 0.001$ is used.)

G-differentiability and Riesz's Representation theorem to derive an extremal problem whose necessary condition coincides with the variational inequality formulation of American options.

Among infinite-dimensional optimization problems, the extremal problem we investigated has a few distinctive properties: (1) the domain set of the de-

Fig. 1. The result for an American call option when $E = 100$, $r = 0.03$, $\sigma = 0.2$, and $\delta = 0.07$.

Fig. 2. The result for an American call option when $T = 0.5$, $E = 100$, $r = 0.03$, $\sigma = 0.2$, and $\delta = 0.07$.

cision variable is defined by boundary conditions and an inequality, which is called an obstacle in traditional engineering problems, (2) the objective functional involves parabolic partial derivatives, and (3) the evaluation of the objective needs an integration from $-\infty$ to $+\infty$. These properties make a numerical approach to the solution difficult. These difficulties were overcome by iterative projections onto the domain, the Crank-Nicolson finite-difference approximations, and a finite length sub-interval approximation, respectively. We discovered that, when compared with binomial methods and projective successive over relaxation methods, the proposed gradient projection method gives fast and accurate solutions for several different American call options. In addition the gradient projection method with finite difference approximations provides an easy-to-implement numerical tool for solving American option pricing problems.

We conclude this paper by noting that the gradient projection method presented is also applicable to more complicated models of multi-asset options where underlying asset values are correlated.

Asset Price	Binomial Method	$(L, M) = (20, 100)$		$(L, M) = (50, 300)$	
		Grad Proj	**PSOR**	**Grad Proj**	**PSOR**
80.000	0.219	0.2476	0.2476	0.2134	0.2134
90.000	1.386	1.2855	1.2855	1.3437	1.3437
100.000	4.783	4.5515	4.5515	4.7068	4.7068
110.000	11.098	10.9719	10.9719	11.0389	11.0389
120.000	20.000	20.0382	20.0382	19.9730	19.9730
	Cal Time (sec)	0.0581	0.0816	0.1563	0.2344
	# Total Iterations	161	80	233	310
	Parameter	$\theta_k = 0.03$	$\omega = 1.00$	$\theta_k = 0.01$	$\omega = 1.04$
Asset Price	**Binomial Method**	$(L, M) = (100, 500)$		$(L, M) = (200, 1000)$	
		Grad Proj	**PSOR**	**Grad Proj**	**PSOR**
80.000	0.219	0.2141	0.2142	0.2171	0.2171
90.000	1.386	1.3613	1.3616	1.3773	1.3774
100.000	4.783	4.7415	4.7420	4.7686	4.7687
110.000	11.098	11.0669	11.0672	11.0875	11.0876
120.000	20.000	19.9957	19.9957	19.9975	19.9975
	Cal Time (sec)	0.4063	0.7031	2.6406	3.0781
	# Total Iterations	406	565	1396	1259
	Parameter	$\theta_k = 0.005$	$\omega = 1.09$	$\theta_k = 0.002$	$\omega = 1.24$

Table 3
A Comparison between Gradient Projection Method and PSOR. $T = 0.5$, $E = 100$, $r = 0.03$, $\sigma = 0.20$, and $\delta = 0.07$. (Binomial method with 15,000 steps is used to compare the values.)

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