Conditional Value-at-Risk Model for Hazardous Materials Transportation

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Abstract

This paper investigates how the conditional value-at-risk (CVaR) can be used to mitigate risk in hazardous materials (hazmat) transportation. Routing hazmat must consider accident probabilities and accident consequences that depend on the hazmat types and route choices. This paper proposes a new method for mitigating risk based on CVaR measure. While the CVaR model is popularly used in financial portfolio optimization problems, its application in hazmat transportation is new. A computational method for determining the optimal CVaR route is proposed and illustrated by a case study in the road network surrounding Albany, NY.

1 INTRODUCTION

Accidents involving hazardous materials (hazmat) are well known low-probability, high-consequence incidents. While the probability of hazmat accidents is very low, the consequence can be catastrophic. The U.S. had about 15,000 hazmat accidents in the year 1998 of which 429 were classified as serious accidents Kara and Verter (2004).

We develop a new routing method to determine the safest route for hazmat transportation. Recently proposed value-at-risk (VaR) models for hazmat Kang et al. (2011b,a) provide a flexible decision making framework for routing hazmat. This paper extends the framework to conditional value-at-risk (CVaR) models.

In this paper, we assume that link accident probabilities are known constants, although in real settings the accident probabilities may be random and follow probabilistic distributions. At the same time, however, the data to compute theses probabilities is very limited as hazmat accidents are rare events. Therefore it is best to assume that the accident probabilities are constant. Such an assumption brings a unique feature to the hazmat VaR/CVaR problem which is different from VaR/CVaR in other applications including financial portfolio management. While the risk measure in finance VaR/CVaR problem is normally assumed continuous, the risk measure (accident consequence) in the hazmat VaR/CVaR problem is a discrete random variable.

VaR has been criticized Nocera (2009); Einhorn (2008), especially after the recent financial crisis in 2007-2008. One criticism is that VaR cuts off and ignores what would happen in the tail. The same argument applies to the use of VaR in hazmat transportation. A road segment with very small accident probability but very large accident consequence can be cut off in VaR computations. CVaR however has better behavior in long tail accounting for losses exceeding VaR Sarykalin et al. (2008). This provides a motivation for this paper.

However, for avoiding the most catastrophic accident consequence, goverment interventions such as curfews and road bans may be better approaches. Suppose that the accident probability when traversing Wall Street in New York City is very small; a hazmat accident consequence on this same street segment is undoubtedly large. It is quite obvious that we do not want hazmat trucks traveling frequently on Wall Street. Therefore restrictions are better than any other routing methods in such cases. Before determining the hazmat route, we should exclude such road segments with the highest consequences. It is noted, however, that the CVaR models can avoid the use of large-consequence links by choosing a sufficiently large confidence level that represents extreme risk-averse attitude.

Compared to VaR, CVaR has better mathematical and computational properties, in addition to the better behavior in long tail. CVaR optimization problems are in general easier to solve, as CVaR problems are convex Rockafellar and Uryasev (2000). This is also true for hazmat applications of CVaR, while additional computational complexity is added: CVaR optimization in hazmat transportation is a convex *discrete* optimization.

The rest of this paper is organized as follows. We provide the CVaR model in Section 2, and a computational method in Section 3. A numerical example in Albany, NY is provided in Section 4. We conclude this paper in Section 5.

2 CONDITIONAL VALUE-AT-RISK MODEL

We consider a graph $G(\mathcal{N}, \mathcal{A})$ and an origin-destination pair. Suppose that we have some estimates of hazmat accident probability and accident consequence, denoted by p_{ij} and c_{ij}, respectively, in each road segment (i, j). The expected consequence of a hazmat truck traveling along path l is as follows Alp (1995):

$$R_l = \sum_{(i_k,j_k)\in\mathcal{A}_l} \prod_{(i_h,j_h)\in\mathcal{A}_l, h<k} (1 - p_{ij}) p_{ij} C_{ij} \tag{1}$$

where $\mathcal{A}_l$ is the set of all arcs in path l, and (i_k, j_k) is the k-th arc in path l. The expression (1) assumes that the shipment terminates once an accident happens in any road segment. It is noted that accident probabilities p_{ij} are extremely small, usually in the range of 10^{-8} to 10^{-6} per mile traveled Abkowitz and Cheng (1988). Therefore we can approximate as

$$\prod_{(i_h,j_h)\in\mathcal{A}_l, h<k} (1 - p_{ij}) \approx 1$$

Consequently, we obtain the following approximation Jin and Batta (1997):

$$R_l \approx \sum_{(i,j)\in\mathcal{A}_l} p_{ij} C_{ij} \tag{2}$$

Each path l consists of a set of arcs $\mathcal{A}_l$. Let $C^l_{(k)}$ denote the k-th smallest value in the set $\{C_{ij} : (i, j) \in \mathcal{A}_l\}$, and $p^l_{(k)}$ the corresponding arc accident probability. Then the risk measure R_l has the following probabilistic values:

$$R_l = \begin{cases} 0, & \text{w.p.} \quad 1 - \sum_{i=1}^{m_l} p^l_{(i)} \\ C^l_{(1)}, & \text{w.p.} \quad p^l_{(1)} \\ \vdots \\ C^l_{(m_l)}, & \text{w.p.} \quad p^l_{(m_l)} \end{cases} \tag{3}$$

where m_l denotes the cardinality of the set $\mathcal{A}_l$ and w.p. stands for "with probability". Given R_l as above, the cumulative distribution function (CDF) of R_l can be derived as follows:

$$F_{R_l}(r) = \Pr(R_l \leq r) = \begin{cases} 1 - \sum_{i=1}^{\overline{m}^l} p^l_{(i)}, & \text{if } r \leq 0 \\ 1 - \sum_{i=2}^{\overline{m}^l} p^l_{(i)}, & \text{if } 0 < r \leq C^l_{(1)} \\ \vdots \\ 1 - \sum_{i=k+1}^{\overline{m}^l} p^l_{(i)}, & \text{if } C^l_{(k-1)} < r \leq C^l_{(k)} \\ \vdots \\ 1, & \text{if } C^l_{(\overline{m}^l)} < r \end{cases} \tag{4}$$

We denote the set of all paths in the network by $\mathcal{P}$. The VaR for path $l \in \mathcal{P}$, with the confidence level α, is defined as:

$$VaR^l_\alpha = \min\{r : \Pr(R_l \leq r) \geq \alpha\} \tag{5}$$

For a path $l \in \mathcal{P}$ at the confidence level α, the CVaR is defined as:

$$CVaR^l_\alpha = \frac{1}{\alpha} \int_0^\alpha VaR^l_\beta d\beta \tag{6}$$

which is in the form of the expected shortfall Acerbi (2002). For continuous random variables, CVaR equals the conditional expectation of risk that exceeds VaR. However, for discrete random variables as in this paper, a more general definition such as (6) must be used Sarykalin et al. (2008).

CVaR in the form (6) is hard to be considered in an optimization problem format because VaR^l_β and its integration are not available in analytical form. Following Rockafellar and Uraysev(2000), we consider the following function:

$$\Phi^l_\alpha(v) = v + \frac{1}{1-\alpha}\mathbb{E}[R_l - v]^+ \tag{7}$$

$$\approx v + \frac{1}{1-\alpha} \sum_{(i,j)\in\mathcal{A}^l} p_{ij}[C_{ij} - v]^+ \tag{8}$$

where we denote $[x]^+ = \max(x, 0)$. Then, we can show that the CVaR minimization is equilvalent to minimize Φ_α^l by choosing a path $l \in \mathcal{P}$ at the confidence level α. That is,

$$\min_{l \in \mathcal{P}} CVaR_\alpha^l = \min_{l \in \mathcal{P}, v \in \mathbb{R}^+} \Phi_\alpha^l(v) \tag{9}$$

We can write

$$\min_{l \in \mathcal{P}, v \in \mathbb{R}^+} \Phi_\alpha^l(v) = \min_{v \in \mathbb{R}^+} \min_{l \in \mathcal{P}} \Phi_\alpha^l(v) \tag{10}$$

$$= \min_{v \in \mathbb{R}^+} \left(v + \frac{1}{1-\alpha} \min_{l \in \mathcal{P}} \sum_{(i,j) \in \mathcal{A}^l} p_{ij}[C_{ij} - v]^+ \right) \tag{11}$$

$$= \min_{v \in \mathbb{R}^+} \left(v + \frac{1}{1-\alpha} \min_{l \in \mathcal{P}} \sum_{(i,j) \in \mathcal{A}^l} m_{ij}(v) \right) \tag{12}$$

where we defined

$$m_{ij}(v) \equiv p_{ij}[C_{ij} - v]^+ \tag{13}$$

We note that the inner minimization problem is a shortest-path problem with arc travel cost $m_{ij}(v)$. Therefore we can write

$$\min_{v \in \mathbb{R}^+} \left(v + \frac{1}{1-\alpha} \min_{x \in \Omega} \sum_{(i,j) \in \mathcal{A}} m_{ij}(v) x_{ij} \right) \tag{14}$$

where we defined

$$\Omega \equiv \Big\{ x : \sum_{i=O, j \in \mathcal{N}} x_{ij} = 1, \sum_{i \in \mathcal{N}, j = D} x_{ij} = 1, \sum_{j \in \mathcal{N}} x_{ij} = \sum_{k \in \mathcal{N}} x_{ki} \quad \forall i \in \mathcal{N}, \text{ and } x_{ij} \in \{0, 1\} \quad \forall (i,j) \in \mathcal{A} \Big\} \tag{15}$$

3 ALGORITHM

In this section, we propose an algorithm for solving the problem (14). Let us define

$$z_\alpha(v) \equiv \min_{x \in \Omega} \sum_{(i,j) \in \mathcal{A}} m_{ij}(v) x_{ij} = \frac{1}{1-\alpha} \sum_{(i,j) \in \mathcal{A}} p_{ij}[C_{ij} - v]^+ x_{ij} \tag{16}$$

Then, the problem (14) is written as

$$\min_{v \in \mathbb{R}^+} Z_\alpha(v) = (v + z_\alpha(v)) \tag{17}$$

We observe that $z(v)$ is a monotonically nonincreasing function of v. That is

$$(v_1 - v_2)[z_\alpha(v_1) - z_\alpha(v_2)] \leq 0 \tag{18}$$

for all $v_1, v_2 \geq 0$.

We also note that $z_\alpha(v)$ decreases linearly as v increases within each interval of $[C_{(k)}, C_{(k+1)}]$ for all $(k) \in \mathcal{A}$. The rate of linear decrease becomes slower (eventually zero), since the number of arcs with nonzero cost decreases (eventually zero). Therefore the shape of $z_\alpha(v)$ is piecewise linear and convex. Consequently, we conclude that $Z_\alpha(v) = v + z_\alpha(v)$ is a convex function of v.

The problem (17) is an one-dimensional convex optimization problem with the nonnegativity constraint. Therefore, the problem can be solved by any line-search technique, with an efficient shortest-path algorithm like Dijkstra's algorithm for the evaluations of $z_\alpha(v)$.

We propose the following Dichotomous Search Method Bazaraa et al. (2006).

Step 0. Choose two small constants $\varepsilon > 0$ and $L > 0$. Let $[a_1, b_1] = [0, \max(C_{ij} : (i,j) \in \mathcal{A})]$. Set $k = 1$.

Step 1. If $b_k - a_k < L$, stop; the minimum point v^* is obtained in the interval $[a_k, b_k]$. Otherwise compute

$$\lambda_k = \frac{a_k + b_k}{2} - \varepsilon, \qquad \mu_k = \frac{a_k + b_k}{2} + \varepsilon$$

For each $v = \lambda_k$ and $\mu = b_k$, solve the shortest path problem

$$z_\alpha(v) = \min_{x \in \Omega} \sum_{(i,j) \in \mathcal{A}} m_{ij}(v) x_{ij}$$

using Dijkstra's algorithm (or any other efficient algorithm) to obtain $z_\alpha(\lambda_k)$ and $z_\alpha(\mu_k)$.

Step 2. If $Z_\alpha(\lambda_k) < Z_\alpha(\mu_k)$, let $[a_{k+1}, b_{k+1}] = [a_k, \mu_k]$. Otherwise let $[a_{k+1}, b_{k+1}] = [\lambda_k, b_k]$. Update k by $k + 1$, and go to Step 1.

When the algorithm is terminated, we declare the optimal solution is

$$v^* = \frac{a_k + b_k}{2}$$

and the corresponding CVaR value and path are

$$CVaR^*_\alpha = v^* + z_\alpha(v^*) \tag{19}$$

$$x^* = \arg\min_{x \in \Omega} \sum_{(i,j) \in \mathcal{A}} m_{ij}(v^*) x_{ij} \tag{20}$$

respectively

4 NUMERICAL EXAMPLE

In this section, we provide a numerical example of the proposed algorithms in Albany, New York, USA and its nearby highway network. The transportation network considered consists of 46 nodes and 70 arcs as presented in Figure 1 Kang et al. (2011b).

The nominal accident probabilities are computed by $p_{ij} = 10^{-6} \times$ (length of arc (i,j)) Abkowitz and Cheng (1988). The nominal accident consequences C_{ij} are computed using the

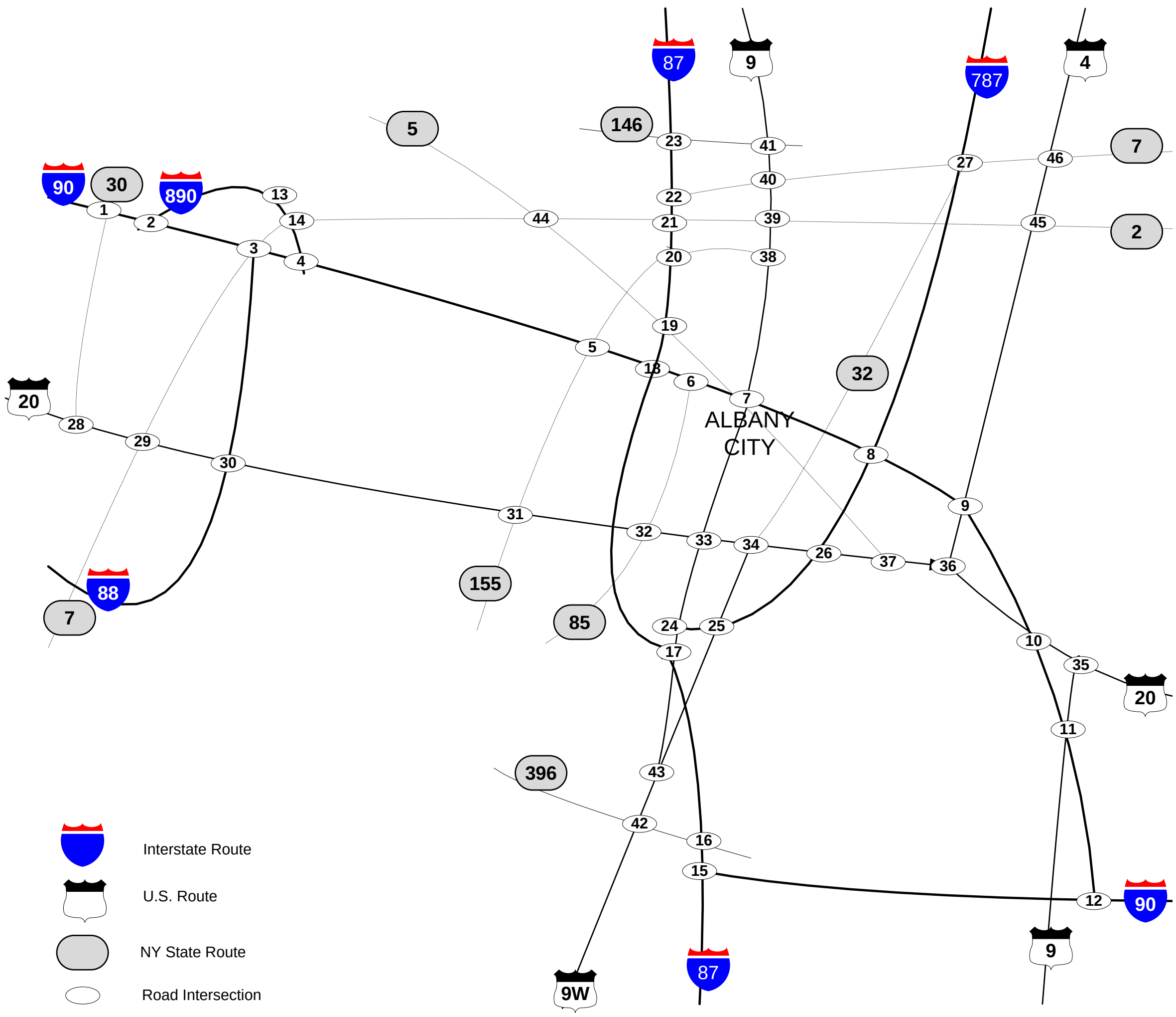

Figure 1: Albany Area Highway Network

Table 1: Example Optimal CVaR Paths

α	Optimal CVaR Path
0.999985	$\{1, 28, 29, 30, 31, 32, 33, 34, 26, 37, 36, 10, 35, 11, 12\}$
0.999993	$\{1, 2, 3, 30, 31, 32, 33, 34, 26, 37, 36, 10, 11, 12\}$
0.999999	$\{1, 2, 3, 4, 5, 6, 32, 33, 34, 26, 37, 36, 10, 11, 12\}$

λ-neighborhood concept Batta and Chiu (1988). The road length and population statistics are obtained from Department of Transportation and Department of Commerce websites.

In Table 1, for various confidence levels, we provide the corresponding optimal CVaR paths obtained by using the algorithm presented in Section 3. We note that each optimal CVaR path remains optimal within a certain interval of α, while only a few distinct examples are provided in Table 1. The CVaR problem for each α is solved within 1 second in a generic personal computer.

5 CONCLUDING REMARKS

We have provided a routing method for hazmat transportation applying the CVaR risk measure. Unlike the VaR optimization model for hazmat transportation, the CVaR model is easier to optimize and the derivation of a numerical algorithm is more straightforward. While the definition of VaR model is intuitively easier to understand its meaning, CVaR better accounts the risk in the long tail.

In this paper, we have assumed that the accident probabilities and accident consequences are known constants. However, in real situations, such data are usually unavailable, which makes the route decision more difficult. In the future research, we intend to study how we may determine a safe route under data uncertainty using robust optimization methods.

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