Reaction Function Based Dynamic Location Modeling in Stackelberg-Nash-Cournot Competition

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Abstract

We formulate a dynamic facility location model for a firm locating on a discrete network. It is assumed that this locating firm will act as the leader firm in an industry characterized by Stackelberg leader-follower competition. The firm's I competitors are assumed to act as Cournot firms and are each assumed to operate under the assumption of zero conjectural variation with respect to their $I-1$ Cournot competitors. Using sensitivity analysis of variational inequalities within a hierachical mathematical programming approach, we develop reaction function based dynamic models to optimize the Stackelberg firm's location decision. In the second half of this paper, we use these models to illustrate through a numerical example the enhanced insights yielded by a reaction based, dynamic approach

Keywords: Dynamic Stackelberg equilibrium location modeling, reaction functions

1 Introduction

When a firm locates a new plant, and begins producing and shipping product to markets on a network, this usually stimulates certain reactions on the network. For example, the introduction of a new plant increases the overall capacity of an industry, and hence can perturb the established economic equilibrium of supplies, demands and flows. The introduction of this new capacity, and in the case of an "entering" firm, the introduction of an entirely new competitor on this network, will typically stimulate competitive responses from existing firms in the industry. In general, we can characterize that the dynamics and existing equilibrium of a market or markets will be affected by the location decision of a firm. This suggests that to truly make a profit maximizing location decision, a firm must anticipate the market's reaction to a potential location decision, in its (the firm's) actual location decision-making process. It is this need to anticipate the market's reaction that spawns our interest in developing facility location models that somehow include projected market reactions endogenously within the firm's profit maximizing facility location objective function.

Plant facility location models provide inputs to a decision (i.e., locating a plant) which almost by definition, implies a significant commitment on the part of a firm to a location for a period of years. Thus, the ability to explicitly evaluate the impact of a location decision over a multi-period (multi-year) planning horizon represents an important capability. In particular, the multi-period planning horizon facilitates the evaluation of the proper timing of location decisions, in addition to the determination of the best location(s). Further, this allows the firm's location decision to better meet forecast growth and/or shrinkage in market demand over time. Dynamic models offer this evaluative capability, while static, single period models do not.

In this paper, we formulate a dynamic, reaction function based competitive facility location model (or what we term an equilibrium facility location model - Miller, Friesz and Tobin, 1995) for a firm locating on a discrete network. This formulation extends and enhances previous single period modeling efforts (e.g., see Miller, Tobin and Friesz, 1992). We assume that this locating firm is an "entering" firm (i.e., entering an industry) competing with several other oligopolistic competitors. This does not represent a limiting assumption however, as the proposed model would apply equally as well to an established firm seeking to expand its manufacturing capacity. We further assume that the locating firm will act as the leader firm in an industry characterized by Stackelberg leader-follower(s) type oligopolistic competition (Friedman, 1977). The other I competitors in this industry are assumed to act as Cournot firms who each operate under the Cournot assumption of zero conjectural variation with respect to their $I-1$ Cournot competitors. (That is to say, in making its own production and shipping decisions, each Cournot firm assumes that its "Cournot" competitors will hold their production and shipping activities at existing levels.) We do assume that the I Cournot firms will react to the location/production/shipping activities of the Stackelberg firm. Thus, the Stackelberg firm makes its location, production, and shipping decisions taking into account the reaction of the I Cournot firms to its (the Stackelberg firm's) location, production and shipping decisions. Therefore, the profit-maximizing objective function of the Stackelberg firm's location model includes a Cournot reaction function that projects the anticipated reaction of the Cournot firms to its (the Stackelberg firm's) integrated location/distribution decisions. The Stackelberg firm's profit-maximizing decisions, along with the Cournot-Nash firms' reactions to these decisions, are termed a Stackelberg-Nash-Cournot (SNC) equilibrium.

To model all of the reactions of all of the Cournot firms to the Stackelberg firm's multiple decisions over multiple time periods represents a difficult problem. An approach to this problem is to utilize sensitivity analysis of Cournot-Nash equilibria to develop Cournot reaction functions. This facilitates expressing all of the reactions of all the Cournot firms to the Stackelberg firm as a vector function of a vector (namely, a vector function of the Stackelberg firm's vector of shipping activities.) A key underpinning of this modeling approach is that shipments of the Stackelberg firms to each node of the network are treated as an extraneous supply that affects the price function contained in each Cournot firm's profit-maximizing objective function over all markets. Using sensitivity analysis methods, partial derivatives of the Cournot firms' decisions are generated with respect to the Stackelberg firm's shipments.

The remainder of this paper is organized as follows. To develop a dynamic SNC competitive facility location model requires that we first construct a dynamic Cournot-Nash equilibrium model. In section 2, therefore, we briefly review the formulation of a dynamic Cournot-Nash model. We will observe that this model can be formulated both as a mathematical programming problem and as a variational inequality. This facilitates the development of a multiperiod Cournot reaction function in section 3 based upon sensitivity analysis techniques for variational inequalities. In this section, we employ this reaction function to formulate a dynamic SNC equilibrium facility location model. Section 4 then offers a simple solution algorithm designed to solve the dynamic SNC location model for illustrative problems. In section 5, we present results for a small example dynamic location problem. This example problem illustrates both the importance of anticipating competitor reactions and of using a multiperiod planning horizon in the location decision-making process. In particular, we illustrate that in certain cases a firm can choose a wrong or suboptimal location if it makes a decision without the benefit of a reaction function. Further, our numerical examples will depict the power of a dynamic model to better address the timing and sizing issues of facility location. Section 6 concludes this paper with some final comments on dynamic, reaction based equilibrium facility location models.

2 A Dynamic Cournot-Nash Network Equilibrium Model

The formulation of a dynamic Cournot-Nash network equilibrium model represents the first step in developing our dynamic SNC model. We can construct a dynamic Cournot-Nash model by building upon previous models reported for single period (or static) spatial and aspatial Cournot-Nash models. Examples of previous formulation include those of Lions and Stampacchia (1967), Gabay and Moulin (1980), Murphy, Sherali and Soyster (1982), Harker (1984, 1986), Haurie and Marcotte (1985), Dafermos and Nagurney (1987), Marcotte (1987), Nagurney (1988), Miller, Tobin and Friesz (1991), and Miyagi (1991). The interested reader is also referred to Dockner (1992), Nagurney, Dupuis and Zhang (1994), and Wie and Tobin (1997) for examples of alternative approaches to modeling the dynamics and adjustment processes associated with oligopolistic equilibria.

2.1 Definition and Formulation

We formulate a Cournot-Nash equilibrium model under the standard assumption that there exist I Cournot firms on a discrete network all supplying a homogeneous good in a noncooperative fashion. Each of these competitors is assumed to operate under the Cournot assumption of zero conjectural variation with respect to its $I-1$ competitor firms in the industry. (That is to say, in making its own production and shipping decisions, each Cournot firm assumes that its $I-1$ competitors will hold their production and shipping activities at existing levels.)

To develop the Cournot-Nash model and all subsequent formulations in this paper, we employ the notation in Table 1.

To facilitate a general formulation and to simplify notation, we allow for the possibility that each firm could produce at every node; if a firm i does not have production capability at node l during period t, then $Q_l^{it} = 0$. We also assume that each firm's production facilities have a fixed finite capacity, since we do not allow for fixed costs associated with capacity expansion. Also note that the total production cost function $v^{it}(q^{it})$ and the total transportation cost function $t^{it}(s^{it}, \bar{S}^{it})$ for each firm are general and allow for interactions among production locations and among transportation routes. Such interactions include volume discounts in inputs and shipment consolidations. In addition, the transportation cost functions allow for interactions among the firms' shipments (as would be the case when the transportation system has limited capacity and many of the firms use the same system). The market inverse demand function $\rho_l^t(D_l^t)$ could also be made general allowing for interactions among markets (other than those due to goods shipped among markets). However, since we are considering a single homogeneous product and interactions due to product movement are modeled explicitly, any other interactions are not important to the problem at hand.

With this background, we can now define a dynamic Cournot-Nash equilibrium as a set of non-negative output vectors q^{it*} (one for each $i = 1, ..., I$, for each $t = 1, ..., T$), a set of non-negative sales vectors d^{it*} (one for each $i = 1, ..., I$, for each $t = 1, ..., T$), and a set of non-negative shipping vectors s^{it*} (one for each $i = 1, ..., I$, for each $t = 1, ..., T$) such that for each $i = 1, ..., I$, for each $t = 1, ..., T$; q^{it*}, d^{it*} and s^{it*} are the optimal solution to the problem:

$$\max \; z^i = \sum_{t \in T} \sum_{l \in K} d_l^{it} \rho_l^t \left(d_l^{it} + \bar{D}_l^{it*} \right) - \sum_{t \in T} v^{it} \left(q^{it} \right) - \sum_{t \in T} t^{it} \left(s^{it}, \bar{S}^{it} \right) \qquad (1)$$

$t = [1, ...T]$ denotes the set of all time periods in the T period planning horizon

I denotes the number of profit maximizing Cournot firms, $i = 1, 2, ..., I$

l, j denote nodes of the network

K denotes the set of all nodes of the network, $l = 1, ..., K$

$p_l^t\left(D_l^t\right), D_l^t \geq 0$ represents the inverse demand function at each market (node) $l \in K$during period t,where D_l^t is the total shipments sold (i.e. sales) to node $l \in K$ during period t

q_l^{it} represents the i-th Cournot firm's production output at node l during period t, where $q_l^{it} \geq 0 \quad \forall l \in K$

$q^{it} = \left[q_1^{it}, ..., q_K^{it}\right]$ is the vector of production quantities for the i-th Cournot firm at all nodes of the network during period t.

s_{jl}^{it} represents the i-th Cournot firm's shipments from node j to node l during period t.

$s^{it} = \left[s_{11}^{it}, ..., s_{KK}^{it}\right]$ is the i-th Cournot firm's vector of shipment quantities from all of the K nodes of the network during period t.(Note that the local shipments $s_{ll}^{it}, l = 1, ..., K$ are included in this vector, i.e., it
is assumed that firms must ship their output to their markets [customers] even in the case where production is consumed locally.)

d_l^{it} is the amount sold by the i-th Cournot firm at node l during period t

d_l^t is the vector of shipments from each Cournot firm to node l during period t

$v^{it}\left(q^{it}\right)$ represents the i-th Cournot firm's total cost of producing q^{it}

$t^{it}\left(s^{it}, \bar{S}^{it}\right)$ represents the i-th Cournot firm's total cost to ship s^{it}, where: $\bar{S}^{it} = \sum_{h \in I, h \neq i} s^{it}$

Q_l^{it} is the capacity of the i-th Cournot firm's production facility at node l during period t.

Table 1: Notation For Cournot-Nash Model

subject to

$$q_l^{it} - \sum_{j \in K} s_{lj}^{it} = 0 \qquad \forall l \in K, \text{ for each } t \in T \tag{2}$$

$$d_l^{it} - \sum_{j \in K} s_{jl}^{it} = 0 \qquad \forall l \in K, \text{ for each } t \in T \tag{3}$$

$$q_l^{it} \leq Q_l^{it} \qquad \forall l \in K, \text{ for each } t \in T \tag{4}$$

$$q_l^{it} \geq 0, d_l^{it} \geq 0 \qquad \forall l \in K, \text{ for each } t \in T \tag{5}$$

$$s_{lj}^{it} \geq 0 \qquad \forall l, j \in K, \text{ for each } t \in T$$

$$\bar{D}_l^{it*} = \sum_{h \in I, h \neq i} d_l^{it*}, \qquad \text{for each } l \in K, \text{ for each } t \in T \tag{6}$$

$$\bar{S}^{it*} = \sum_{h \in I, h \neq i} s^{it*}, \qquad \text{for each } t \in T \tag{7}$$

The Cournot-Nash equilibrium model is not of great interest as a stand-alone model for our purposes. In fact, if the parameter values of the coefficients of the inverse demand, production and transportation functions remain the same from one time period to the next (e.g., t to $t+1$), the identical equilibrium solution will obtain for each time period. Nevertheless, there are several important points to consider.

First, note that the equilibrium solutions for each time period $t = 1, ..., T$ are independent of each other. Thus, one can essentially think of problem (1)-(7) as T independent problems. Second, we can formulate the dynamic Cournot-Nash equilibrium model as a variational inequality, or essentially as T variational inequalities, given the independence of each period from all other periods. For purposes of brevity, we simply state this and refer the reader to Miller, Tobin and Friesz (1991), and Tobin, Miller and Friesz (1995) for detailed discussions of equivalent variational inequality formulations of Cournot-Nash equilibrium models. Additionally, these citations will provide the interested reader with detailed background on both the development of, and key characteristics of Cournot-Nash static and dynamic models - discussions that we again omit in this paper for purposes of brevity. Because we can formulate problem (1)-(7) as a variational inequality, methods for sensitivity analysis of variational inequalities can be applied for each individual time period t. We will observe that this facilitates the development of an independent Cournot reaction function for each time period $t \in T$.

3 Dynamic Stackelberg Profit Maximizing Location Model

The Stackelberg firm must choose its locations, production levels and shipping levels for each time period t taking into account the reactions of the I Cournot firms to these decisions in each separate time period. Thus, the Stackelberg firm's total profit maximizing facility location objective function must include a Cournot reaction function for each time period t. To construct a Cournot reaction function and then a Stackelberg location model, we require the additional notation shown in Table 2.

Briefly, to develop the Cournot reaction functions, we treat the Stackelberg firm's market supplies as parameters in the Cournot-Nash equilibrium model. This is accomplished by including the Stackelberg firm's supplies to market l, d_l^x, in the price function at market l. Thus, we must restate the Cournot-Nash objective function (1) as follows:

$$\sum_{t\in T}\sum_{l\in K} d_l^{it}\rho_l^t\left(d_l^{it}+\bar{D}_l^{it*}+d_l^{xt}\right)-\sum_{t\in T}v^{it}\left(q^{it}\right)-\sum_{t\in T}t^{it}\left(s^{it},\bar{S}^{it}+s^{xt}\right) \qquad (8)$$

If, for all feasible Stackelberg market supplies, the corresponding Cournot-Nash equilibrium problem [(1)-(7), with objective function (8) replacing (1)], has a unique solution; then the Cournot-Nash equilibrium quantities can be considered as a function of the Stackelberg market supplies. The interested reader is referred to Miller, Tobin and Friesz (1991, 1992) and Miller, Friesz and Tobin (1995) for a detailed explanation of this. To conclude this discussion briefly, however, we note that the Cournot-Nash equilibrium can then be represented as a variational inequality that is parametric in (d^{xt}, s^{xt}) and the solution to this variational inequality defines the implicit functions $d_l^{it*}(d^{xt}, s^{xt})$ and $s_{jl}^{it*}(d^{xt}, s^{xt})$. An aggregate Cournot reaction function can now be defined for each node $l = 1, ..., K$. Specifically, define

$$R_l^t\left(d^{xt},s^{xt}\right)=\sum_{i\in I} d_l^{it*}\left(d^{xt},s^{xt}\right) \qquad \forall l\in K$$

as the aggregate sales reaction function, at a node l during period t, of the I Cournot firms to the total shipments to all nodes (markets) $l \in K$ during period t by the Stackelberg firm, and define

$$T_{jl}^t\left(d^{xt},s^{xt}\right)=\sum_{i\in I} s_{jl}^{it*}\left(d^{xt},s^{xt}\right) \qquad \forall j,l\in K$$

as the aggregate transportation reaction function on link j, l of the I Cournot firms to the total shipments to all nodes by the Stackelberg firms during period t, where $T^t(d^{xt}, s^{xt})$ denotes the vector $\left[T_{jl}^t(d^{xt}, s^{xt})\right]$.

The reaction functions are implicit functions defined by the equivalent parametric variational inequality that includes the Stackelberg supplies as parameters. The derivatives of these reaction functions are used to approximate the reaction function locally to obtain a solution algorithm for solving the Stackelberg profit maximizing problem. These derivatives are derived using sensitivity analysis methods for variational inequalities (Dafermos, 1988; Kyparisis, 1987, 1989; Pang, 1988; Qiu and Magnanti, 1989; and Tobin 1986). These methods yield the derivatives of the solutions to the variational inequality with respect to problem parameters.

x_l^t represents the Stackelberg firm's output at node l during period t

$x^t = [x_1^t, ..., x_K^t]$ is the vector of production quantities for the Stackelberg firm at all nodes of the network during period t

s_{jl}^{xt} represents the Stackelberg firm's shipments from node j to node l during period t

$s^{xt} = [s_{11}^{xt}, ..., s_{KK}^{xt}]$ is the Stackelberg firm's vector of shipment quantities from all K nodes of the network during period t

$v^t(x^t)$ represents the Stackelberg firm's total cost of producing x during period t

$t^t\left(s^{xt}, \bar{S}^{xt}\right)$ represents the Stackelberg firm's total cost to ship s^{xt}, where:

$\bar{S}^{xt} = \sum_{i \in I} s^{it}$

d_l^{xt} is the amount sold by the Stackelberg firm at node l during period t

d^{xt} is the Stackelberg firm's vector of total amounts shipped (i.e., sold) to each market during period t

Q_l^{xt} is the capacity of the Stackelberg firm's production facility at node l during period t

$\bar{Q}^t$ is the maximum amount of new production which the Stackelberg firm may locate (and/or have) over the entire network during period t

F_l^t is the portion of the total fixed location cost of establishing a production facility at node l allocated to period t

y_l^t is a discrete location decision variable; $y_l^t = 1$ if the Stackelberg firm locates a production facility at node l during period t or has located at node l during a previous periods, $y_l^t = 0$ otherwise.

Table 2: Notation For Stackelberg Location Model

We can now define a dynamic Stackelberg profit maximizing equilibrium facility location model. A set of Stackelberg facility location vectors y^{t*} (one for each $t = 1, ..., T$), a set of $(T) \times (I+1)$ non-negative output vectors $\left[x^{t*}; q^{1t*}, ..., q^{It*}\right]$ (one $I+1$ set for each $t = 1, ..., T$), a set of $(T) \times (I+1)$ non-negative shipping vectors $\left[s^{xt*}; s^{1t*}, ..., s^{It*}\right]$ (one $I+1$ set for each $t = 1, ..., T$), and a set of $(T) \times (I+1)$ non-negative sales vectors $\left[d^{xt*}; d^{1t*}, ..., d^{IT}\right]$ (one $I+1$ set for each $t = 1, ..., T$), represents a dynamic Stackelberg-Nash-Cournot equilibrium solution if x^{t*}, s^{xt*}, and y^{t*}, solve the following problem:

$$\max \ z^x = \sum_{t\in T}\sum_{l\in K} d_l^{xt}\rho_l^t\left(d_l^{xt} + R_l^t\left(d^{xt}, s^{xt}\right)\right) - \sum_{t\in T} v^t\left(x^t\right) - \sum_{t\in T}\sum_{l\in K} F_l^t y_l^t - \sum_{t\in T} t^t\left(s^{xt}, T^t\left(d^{xt}, s^{xt}\right)\right) \tag{9}$$

subject to

$$x_l^t - \sum_{j\in K} s_{lj}^{xt} = 0 \qquad \forall l \in K, \text{ for each } t \in T \tag{10}$$

$$d_l^{xt} - \sum_{j\in K} s_{jl}^{xt} = 0 \qquad \forall l \in K, \text{ for each } t \in T \tag{11}$$

$$x_l^t \leq Q_l^{xt} y_l^t \qquad \forall l \in K, \text{ for each } t \in T \tag{12}$$

$$\sum_{l\in K} Q_l^{xt} \leq \bar{Q}^t \text{ for each } t \in T \tag{13}$$

$$y_l^t = (0, 1) \qquad \forall l \in K, \text{ for each } t \in T \tag{14}$$

$$y_l^{t+1} \geq y_l^t \qquad \forall l \in K, \text{ for each } t \in T \tag{15}$$

$$x_l^t \geq 0, \ d_l^{xt} \geq 0 \qquad \forall l \in K, \text{ for each } t \in T \tag{16}$$

$$s_{lj}^{xt} \geq 0 \qquad \forall l, j \in K, \text{ for each } t \in T$$

and if, for each $i = 1, ..., I$; for each $t = 1, ..., T$; $q^{it*}, s^{it*}, d^{it*}$ are optimal solutions to (1) through (7) (i.e., the dynamic Cournot-Nash equilibrium problem for each Cournot firm i, given a vector of Stackelberg market supplies).

In the dynamic Stackelberg location model, constraints (10) and (11) guarantee that in each time periods t, the Stackelberg firm does not ship more from a node l than it produces at that node, and that it does not sell more at a node l than it ships to that node. Constraint (12) assures that the firm's production at a node l does not exceed its capacity Q_l^{xt} if a facility is open at l in time period t (i.e., $y_l^t = 1$). Constraint (13) limits the Stackelberg firm's total level of production at all nodes in each time period t, while constraint (14) restricts the location decision variables y to be zero or one. Constraint (15) plays a key role in that it links the Stackelberg firm's location decisions over the planning horizon $t = 1, ..., T$. Specifically, (15) assures that once the Stackelberg firm opens a facility at node l in time period t, this facility will remain open for all subsequent time periods in the planning horizon. Equations (16) represent the standard non-negativity constraints for the model's decision variables. Finally, note that the revenues and costs defined in the dynamic model represent appropriately discounted flows over time which yield a net present value of the Stackelberg firm's profit over the planning horizon.

Note also that similar to the Cournot firms' cost functions, both the Stackelberg firm's total production cost function $v^t(x^t)$ and transportation cost function $t^t(s^{xt}, T^t(d^{xt}, s^{xt}))$ for each period t are general. Thus, the functions permit individual interactions among the individual production locations of the Stackelberg firm and among the transportation routes of all firms in a time period t. Theoretically, one could also allow interactions among these cost functions over time periods $t \in T$, if a compelling rationale existed to do so. We, however, assume no interactions between cost functions in different time periods in (9)-(16).

This dynamic facility location model in Stackelberg-Nash-Cournot equilibrium offers several attractive features. First, as noted, in contrast to a static version of this model (Miller, Tobin, and Friesz, 1992), the dynamic model addresses the timing of location decisions in addition to simply the actual location choice. Further, one can easily modify the dynamic model to evaluate such alternatives as staged capacity location (as we will illustrate later) and expansion (see Miller, Friesz, and Tobin, 1995). Such an extension for example, allows a firm to determine if it is more profitable in certain cases to locate and then expand its capacity at a site over time in response to increasing market demands, rather than to initially construct a large facility.

The multi-period dimension of the dynamic model also makes it possible to change any and/or all of the model's functional forms from one time period t to the next. This provides considerable modeling flexibility. For example, market demands vary over time and particularly in fast-changing markets, the ability to evaluate demand over multiple periods can prove beneficial. Similarly, a firm's costs also tend to vary over time. A firm often experiences a learning curve in its manufacturing operations, particularly when locating new capacity. This can create a situation in which variable production costs decrease over time while the firm moves along this learning curve. Transportation costs also may change in response to supply and demand relationships, congestion, etc. In summary, a dynamic Stackelberg location model can explicitly evaluate all these types of time-related phenomenon. Variation in a firm's cost structure naturally can be modeled for both the Cournot firms and Stackelberg firm as appropriate.

4 Illustrative Solution Approach

In general, the dynamic Stackelberg location problem represents a difficult model to solve. For purpose of this paper, we will limit ourselves to a brief discussion of the algorithmic aspects of this model.

It is interesting to note that the dynamic Cournot-Nash submodel of the overall Stackelberg model represents a very tractable problem. As previously discussed, from an algorithmic point of view, one can consider each time period $t \in T$ as independent from the other $T - 1$ time periods in the planning horizon for the Cournot-Nash stand-alone problem. Therefore, a block diagonalization algorithmic approach for solving single period variational inequalities (see e.g., Harker, 1984 and 1986; Miller, Tobin and Friesz, 1991 and 1992; and Miller, Friesz and Tobin, 1995) can be extended directly to solve a dynamic Cournot-Nash equilibrium model. This results because the equilibrium Cournot-Nash solution for any period $t \in T$ is not affected by the equilibrium solutions of any of the other $T-1$ periods in the planning horizon. A review of the constraints for problem (9)-(16) illustrates the independence of each period. The individual time periods $t \in T$ of the Stackelberg location submodel (9)-(16) of the overall bilevel, Stackelberg model are also relatively independent of each other. Constraint (15) which forces any facility opened in a period t to remain open for all remaining periods of the planning horizon, represents the only link between time periods. This suggests the possibility of developing a decomposition algorithm to solve the model (9)-(16). This remains a topic for future research.

4.1 ENUMERATION APPROACH

To state illustrative dynamic Stackelberg location model solutions, we will employ an enumeration algorithm. This enumeration algorithm includes the following three core components:

I. Solve the dynamic Cournot-Nash model [Problem (1)] to obtain the equilibrium solution (i.e., the production levels and shipping patterns) for the I Cournot-Nash firms competing on the network, given a set of Stackelberg supplies.

II. Perform sensitivity analysis on the equilibrium solution(obtained in Step I) and create a linear approximation to the Cournot-Nash reaction function (based on sensitivity analysis).

III. Solve the nonlinear mathematical optimization programming submodel [Problem (9), with a linear reaction function] to obtain a new approximation of the Stackelberg firm's profit maximizing location solution. Note that the Stackelberg firm's objective function contains the Cournot reaction function created in Step II. (Repeat these steps until the defined convergence criteria is satisfied.)

The individual steps required to solve the dynamic Stackelberg location problem by explicit enumeration are as follows:

Step 0. Determine all combinations of locations (and the timing) to be enumerated.

Step 1. Pick a Stackelberg locational pattern to be evaluated, and set $y_l^t = 1$ for each node l included in this pattern, for each period $t \in T$ when a node l will have a producing facility.

Step 2. Choose an initial value for s_{jl}^{it} $\forall j, l \in K$, $\forall t \in T$, $\forall i \in I$, and choose an initial value for s_{jl}^{xt} $\forall j, l \in K$, $\forall t \in T$. Set counters $z = 0$ and $w = 0$.

Step 3. Set $z = z + 1$, and solve the mathematical programming problem (1) for each firm $i \in I$, thereby obtaining the distribution pattern represented by s_{jl}^{itz} $\forall j, l \in K$, $\forall t \in T$, $\forall i \in I$.

Step 4. If $z \leq 1$, return to Step 3, else if $\left| s_{jl}^{itz} - s_{jl}^{itz-1} \right| \leq \varepsilon$ $\forall j, l \in K$, $\forall t \in T$, $\forall i \in I$, where ε is a predetermined tolerance; then the current solution is a Cournot-Nash equilibrium solution – Go to Step 5. If this is not true, return Step 3.

Step 5. Calculate the derivatives of the Cournot-Nash solution with respect to the Stackelberg decision quantities to determine how each of the I Cournot-Nash firms would react to an increase in the Stackelberg firm's shipments. (See Miller, Tobin and Friesz, 1991 for a discussion of how to determine these derivatives.)

Step 6. Estimate the Cournot-Nash reactions by forming linear approximations utilizing the derivatives developed in Step 5. These linear approximations express the change in a Cournot-Nash firm's shipments resulting from extraneous changes in the Stackelberg shipments.

Step 7. Set $w = w + 1$. Solve the nonlinear mathematical programming program (9). If $w \leq 1$, return to Step 3. Else if, $\left| s_{jl}^{itw} - s_{jl}^{itw-1} \right| \leq \varepsilon$ $\forall j, l \in K$, $\forall t \in T$, (where ε is a predetermined tolerance); then the current solution is a Stackelberg-Nash-Cournot equilibrium solution for the locations y chosen in Step 1 and it provides the profit maximizing solution to the Stackelberg firm's problem (for this particular location vector y). Stop.

Step 8. Record the profit (and other appropriate data) associated with this enumerated locational pattern. If all locational alternatives have been enumerated, stop. Else, return to Step 1.

For a detailed discussion of the solution issues and characteristics of this enumeration algorithm, the reader is referred to Miller, Tobin and Friesz (1991 and 1992) and Miller, Friesz, Tobin (1995).

5 ILLUSTRATIVE NUMERICAL EXAMPLE

In this section we present a numerical example of the model discussed in Section 3. By means of several alternative location decision models, we illustrate the potential benefits which a reaction function based equilibrium facility location modeling approach can provide. Specifically, we compare the results of five different formulations of a location problem, all based on the same basic data set, to depict how the inclusion or exclusion of reaction functions can affect the quality of the solutions obtained. Similarly, we use these examples to demonstrate the potential impact on solutions created by using a dynamic rather than a static modeling approach.

The location problems are three period problems based on a network consisting of four nodes, with a market at each node; and 16 separate transportation links, one in each direction between each pair of nodes. Two existing firms currently compete on this network (Firms 1 and 2), and each firm has production facilities at all four nodes. A third firm (Firm 3) is locating a plant on this network and entering the industry. The respective inverse demand, production and transportation cost functions of the firms have the following forms:

$$\rho_l^t \left(D_l^t \right) = \alpha_l^t - \beta_l^t D_l^t \tag{17}$$

$$v^{it} \left(q_l^{it} \right) = \sum_l 0.5 c_l^{it} \left(q_l^{it} \right)^2 \tag{18}$$

$$t^{it} \left(s^{it} \right) = \sum_j \sum_i 0.5 t_{jl}^t \left(s_{jl}^{it} \right)^2 \tag{19}$$

Where $\alpha_l^t, \beta_l^t, c_l^{it}$ and t_{jl}^{it} are constants. In what follows, we will compare the optimal production levels and predicted profits determined by a location model to those actually resulting after Firm 3 locates the plant and is competing on the network. The location decision models for Firm 3 that we consider are:

Model 1: Firm 3 employs a standard location model in which the profit function for the locating firm in this model assumes fixed prices and does not include any market reaction. The market demand functions in model 1 are the same in all three periods of the planning horizon.

Model 2: Firm 3 models the market using a Cournot-Nash game theoretic oligopolistic equilibrium model for each location possibility, and chooses the most profitable location in equilibrium. The market demand functions remain the same in all three periods.

Model 3: Firm 3 employs a dynamic Stackelberg "equilibrium facility location model" in which the profit function in the model includes the reactions of the existing market to Firm 3's location and production decisions. Again the market demand functions remain constant over the planning horizon.

Model 4: Firm 3 employs the same model as in model 3, however, the market demand function increases in each time period.

Model 5: As in models 3 and 4, Firm 3 employs a dynamic Stackelberg equilibrium facility location model. However, Firm 3's model in this case allows the firm to consider a "staged" capacity location approach. Additionally, the market demand functions increase from period to period in this model.

In these alternative models, Firm 3 is deciding whether to locate production facilities at node 1 or node 2. (We limit Firm 3's location decision to two nodes for illustrative purposes.) Appendix A displays the coefficients for the demand, production and transportation cost functions, as well as for Firm 3's location cost data.

Profits $(millions)			Production $(thousands)		
Firm 1	Firm 2	Total Industry	Firm 1	Firm 2	Total Industry
53.8	56.4	110.2	3,006	3,194	6,200

Table 3: Cournot-Nash Equilibrium For Existing Duopoly (Prior to Firm 3's Entry)

	Locate At Node 1	Locate at Node 2
Profits $(millions)	41.5	40.4
Production (thousands)	2,277	2,217

Table 4: Firm 3's Predicted Profits and Production Levels Using Model 1

We begin the location modeling illustration by first evaluating the existing market conditions prior to the entry of Firm 3. Table 3 displays the equilibrium profits and production levels of Firms 1 and 2 competing in a duopoly. Note that industry profits prior to Firm 3's entry are $110.2 million over 3 periods. (Our illustrative dynamic problems are 3 period problems, and therefore, all results are stated in terms of 3 period profit and production levels.) To simplify the presentation, in this case and all subsequent cases, we show each firm's total production, but do not present optimal shipment levels to each of the four markets. It should be noted, however, that for any production level, a firm can have multiple shipment patterns with different profit levels resulting from each pattern.

Table 4 shows the "predicted" profits and production levels which Firm 3 expects to generate when it bases its location decision on model 1. This model indicates that Firm 3 should locate at node 1. Recall that in model 1 Firm 3 does not anticipate any market changes in response to its entry - i.e., it does not account for either competitor reactions or changes in equilibrium prices. However, Firm 3's entry will perturb the market, and Table 5 indicates how the market will settle after Firm 3 begins producing at its initially planned production levels based on model 1. Briefly, it turns out that Firm 3 cannot make a profit at its planned production levels, while Firms 2 and 3 throttle back production and see their profits diminish. Again, Firm 3's negative profits result because it has not accounted for either the price elasticity of demand or the reaction of the other firms.

Once Firm 3 begins actual production and distribution, it realizes that its

Firm 3 Locates	Profits $(millions)			Production $(thousands)		
At Node	Firm 1	Firm 2	Firm 3	Firm 1	Firm 2	Firm 3
1	33.2	35.4	negative	2,747	2,918	2,277
2	33.5	35.8	negative	2,753	2,926	2,217

Table 5: Actual Market After Firm 3 Enters Using Model 1

Firm 3 Locates At Node	Profits $(millions)			Production $(thousands)		
	Firm 1	Firm 2	Firm 3	Firm 1	Firm 2	Firm 3
1	43.5	45.9	22.7	2,867	3,045	1,230
2	43.4	45.8	23.3	2,854	3,032	1,350

Table 6: Three Firm Counot-Nash Equilibrium Results When Firm 3 Locates Using Model 2

initial production and shipment levels are not profitable, let alone optimal with respect to the new production and shipment levels of its competitors. Firm 3, therefore, begins to continually optimize its production and shipment levels in response to its competitors, and thus, acts like a Cournot oligopolistic competitor. For illustrative purposes, we now also assume that Firm 3 begins to account for the price elasticity of demand in its planning process. Table 6 displays the equilibrium which results when all three firms compete as Counot-Nash oligopolists. The profits and production levels generated when Firm 3 locates at node 1 represent the equilibrium which would actually develop because Firm 3 chose node 1 using model 1.

Table 6 also indicates the equilibrium which results when Firm 3 plans it market entry using model 2; namely, when it acts as a Cournot firm right from the start. In this model, Firm 3 will react to changes in the market triggered by its entry, just as any Cournot firm would (i.e., it will continue to re-optimize its production and distribution levels in response to its competitors' adjustments). The Cournot-Nash model indicates that Firm 3 can optimize its profits by locating at node 2. Thus Firm 3 will make a different location decision using model 2 rather than model 1. Note that because location model 2 (and all subsequent models we will review) take into account the adjustments of the firms after Firm 3's entry, the prediction of the location model matches that actual market after Firm 3's entry.

In illustrations 1 and 2, we first examined a method which did not include the reactions of competitors in the location model, and then a method which did not consider competitor reactions as effectively as possible. Specifically, in model 1, Firm 3 does not account for the reaction of competitors to the new production; and therefore, the actual outcome after entry differs significantly from that predicted by the model. In model 2, Firm 3 evaluates the reactions of its competitors to its new production and distribution, however, the firm does not use this information as effectively as possible to optimize its location and production. Because the actions of all firms are consistent with the assumptions of model 2, however, the actual resulting market equilibrium matches that predicted by the model. In the following model, Firm 3 does incorporate the market reaction into the profit function in its optimization model.

In model 3, Firm 3 makes its location decision by modeling itself as the leader firm in Stackelberg-Nash-Cournot competition. Thus, Firm 3 predicts and evaluates the reactions of Firms 1 and 2 to its potential location decision as part of its location selection methodology. This allows Firm 3 to fully exercise

Firm 3 Locates At Node	Profits $(millions)			Production $(thousands)		
	Firm 1	Firm 2	Firm 3	Firm 1	Firm 2	Firm 3
1	37.6	39.9	24.2	2,823	2,999	1,608
2	38.1	40.4	24.0	2,830	3,007	1,545

Table 7: Three Firm Stackelberg-Nash-Cournot Equilibrium Results When Firm 3 Locates Using Model 3

Firms	Before Entry (i.e. Duopoly)	Model		
		1	2	3
Firms 1 and 2	110.2	89.4	89.2	77.5
Firm 3	0	22.7	23.3	24.2
Total Industry	110.2	112.1	112.5	101.7

Table 8: Comparison of Actual Profits in Models 1, 2 and 3 ($ millions)

its profit-making potential. Table 7 illustrates the equilibrium solution generated using this methodology. Because Firm 3 anticipated the reactions of its competitors in its optimization model, the actual outcome in the market after Firm 3 enters the industry mirrors model 3's solution. Note that model 3's solution indicates that Firm 3 should locate at node 1 to maximize its profits. This contrasts with the selection of node 2 recommended by model 2 when Firm 3 acts as a Cournot competitors (i.e., when it reacts to its competitors' reactions rather than anticipates their reactions).

The contrast between the results generated when Firm 3 uses models 1 and 2 compared to those generated when it uses model 2 illustrates the potential power of including reaction functions in location decisions. Figure 1 and Tables 8 and 9 summarize the impact of the entry of Firm 3 on Firms 1 and 2 under these alternative location models. Observe that Firm 3's entry substantially reduces the combined profitability of Firms 1 and 2 from a high of $110.2 million (for three periods) in a duopoly, to a low of $77.5 million when Firm 3 locates as the Stackelberg leader firm. Further, the greater the anticipatory powers of Firm 3, the more the profits of Firms 1 and 2 decline. In model 2, when Firm 3 locates as a Cournot firm and accounts for the other firms' reactions by modeling the equilibrium for each location choice, the combined profits of Firms 1 and 2 decrease significantly to $89.2 million. However, in model 3, when Firm 3 anticipates the reaction of Firms 1 and 2 in advance and explicitly accounts for their reactions in its profit maximization calculations, the collective profits of the first two firms drop the most (to $77.5 million), while Firm 3 maximizes its profits.

Models 1, 2 and 3 have provided a set of numerical examples designed to demonstrate the potential power of including reaction functions and analysis of economic equilibria in facility location models. These simplistic problems have illustrated an example where a locating firm (Firm 3) could only determine its truly optimal location strategy by integrating models of market equilibria,

Firms	Before Entry (i.e. Duopoly)	Model		
		1	2	3
Firms 1 and 2	6,200	5,912	5,886	5,822
Firm 3	0	1,230	1,350	1,608
Total Industry	6,200	7,142	7,236	7,430

Table 9: Comparison of Actual Producttion in Models 1, 2 and 3 ($ thousands)

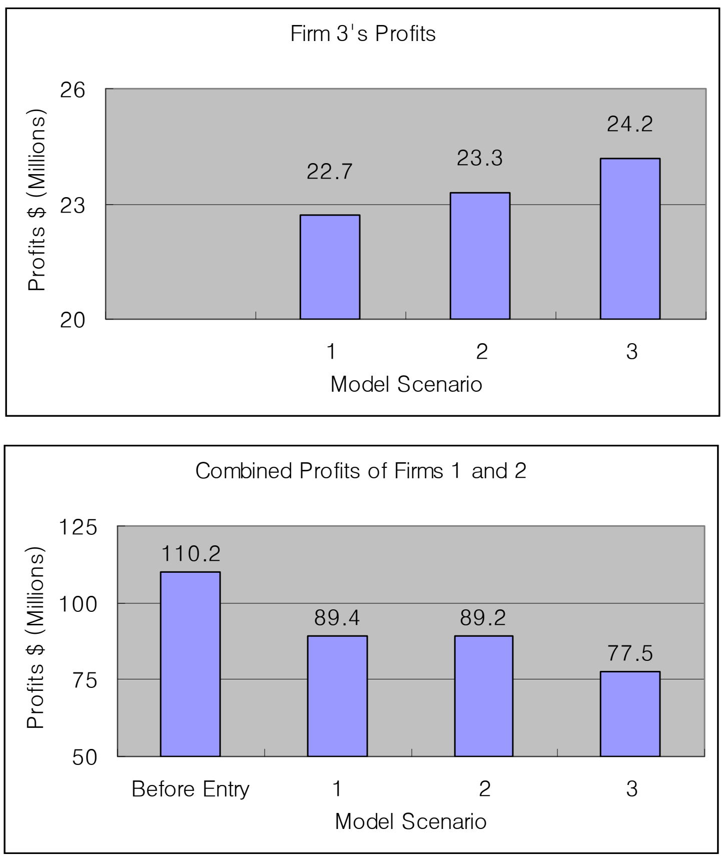

Figure 1: Comparison of Each Firm's Profits Under Alternative Modeling Scenarios

Firm 3 Locates At Node	Profits $(millions)			Production $(thousands)		
	Firm 1	Firm 2	Firm 3	Firm 1	Firm 2	Firm 3
1	78.1	83.3	37.9	4,083	4,296	1,999
2	80.2	85.2	40.4	4,045	4,302	1,969

Table 10: Three Firm Stackelberg-Nash-Cournot Equilibrium Results When Firm 3 Locates Using Model 4 (Demand Increases Each Period)

sensitivity analysis-based reaction functions and models of facility location.

We conclude this numerical section by extending our illustrations to consider Firm 3 facing essentially the same facility location problem, but with the additional caveat that market demand will increase in both periods 2 and 3. Appendix A shows the demand functions for each period. In model 4, Firm 3 is simply determining where to locate a plant which will begin producing in period 1. In model 5, however, the firm will consider both location and timing issues in a staged capacity location problem.

Table 10 indicates that in model 4, Firm 3 can optimize its profits by locating at node 2. This contrasts with the results obtained in model 3 when the demand functions at each node remained constant over the three period planning horizon, and node 1 represented the profit-maximizing alternative for Firm 3. In this case, both the increase and the relative shift in the distribution of demand over time leads to an alternative location decision. These contrasting solutions illustrate the enhanced perspective yielded by a dynamic model relative to a static model. (Model 3 is essentially equivalent to a single period model in that the parameters and variables in each of the three periods are identical in all respects.) In model 4, where Firm 3 can recognize the forecast change in demand over time, it can make a better long run location decision. While the decision in model 3 to locate at node 1 will maximize profit in the short run, in the long run, this represents a suboptimal decision - should demand change over time as suggested in model 4.

To formulate a dynamic Stackelberg facility location and expansion problem (i.e., model 5), we must modify model (9)-(16), and introduce additional notation. Specifically, by replacing constraint (15) with alternative constraints, we will allow the model to consider locating alternative sized facilities either in a single period, or over multiple periods. For example, in model 5, Firm 3 can decide to locate a small facility at a node l during a time period t, and then expand the capacity of the facility at l to that of a large facility later in the planning horizon. Alternatively, Firm 3 can just initially build a large facility at node l. Table 11 provides additional notation to facilitate this example.

In model 5, we assume that Firm 3 incurs greater total fixed location costs if it locates a large facility at a node l in two stages over time rather than if it constructs a large facility initially. This represents a fairly typical example in that it is frequently less expensive to build a facility in one continuous project than to build a smaller facility as one project, and then expand that facility as a second separate project. To formulate a dynamic capacity expansion model

$\overset{S}{y}{}_l^t$ is a discrete location decision variable; $\overset{S}{y}{}_l^t = 1$ if Firm 3 locates, or has previously located and operates, a small production facility at node l during period t, $\overset{S}{y}{}_l^t = 0$ otherwise

$\overset{L}{y}{}_l^t$ is a discrete location decision variable; $\overset{L}{y}{}_l^t = 1$ if Firm 3 locates, or has previously located and operates, a large production facility at node l during period t, $\overset{L}{y}{}_l^t = 0$ otherwise

$\overset{E}{y}{}_l^t$ is a discrete location decision variable; $\overset{E}{y}{}_l^t = 1$ if Firm 3 expands, or has previously expanded and operates, an originally small production facility at node l during period t, $\overset{E}{y}{}_l^t = 0$ otherwise

$\overset{S}{F}{}_l^t$ is the portion of the total fixed location costs of establishing a small production facility at node l allocated to period t (i.e., an amortized cost)

$\overset{L}{F}{}_l^t$ is the portion of the total fixed location costs of establishing a large production facility at node l allocated to period t. Note that $\overset{L}{F}{}_l^t > \overset{S}{F}{}_l^t$ for any particular node l at period t.

$\overset{E}{F}{}_l^t$ is the portion of the total fixed location costs of establishing a small production facility and later expanding it to a large production facility at node l, allocated to period t. That is, the allocated total fixed costs represent the sum of the fixed costs of locating both the initial small production facility and the expansion portion of the facility. Note that $\overset{E}{F}{}_l^t > \overset{L}{F}{}_l^t > \overset{S}{F}{}_l^t$ for any particular node l and period t.

Table 11: Notation for Stackelberg Expansion Problem

which incorporates this cost assumption, we modify problem (9)-(16) by defining three plant or facility types (Table 11). Facility type $\overset{S}{y}{}_l^t$ represents a “small” plant, facility type $\overset{L}{y}{}_l^t$ represents a “large” plant, and facility type $\overset{E}{y}{}_l^t$ represents a “large” plant which was initially a “small” plant and then was expanded sometime after its initial construction. Note again that the allocated period fixed location costs, $\overset{E}{F}{}_l^t$, of facility type $\overset{E}{y}{}_l^t$ exceed those of facility type $\overset{L}{y}{}_l^t$ reflecting the greater total cost associated with constructing a facility in two dependent stages. We can now replace constraint (15) in our original dynamic Stackelberg model (9)-(16) with the following three constraints:

$$\left(\overset{S}{F}{}_l^{t+1}\overset{S}{y}{}_l^{t+1} + \overset{L}{F}{}_l^{t+1}\overset{L}{y}{}_l^{t+1} + \overset{E}{F}{}_l^{t+1}\overset{E}{y}{}_l^{t+1}\right) \geq \left(\overset{S}{F}{}_l^{t}\overset{S}{y}{}_l^{t} + \overset{L}{F}{}_l^{t}\overset{L}{y}{}_l^{t} + \overset{E}{F}{}_l^{t}\overset{E}{y}{}_l^{t}\right) \qquad \forall l \in K, \text{ for each } t \in T \tag{20}$$

where for $t = T + 1$ (i.e., past the planning horizon), set $\overset{E}{y}{}_l^t = 1 \; \forall l \in K$.

$$\overset{S}{y}{}_l^t + \overset{L}{y}{}_l^t + \overset{E}{y}{}_l^t \leq 1 \qquad \forall l \in K, \text{ for each } t \in T \tag{21}$$

$$\overset{S}{y}{}_l^t + \overset{L}{y}{}_l^t \leq 1 \qquad \forall l \in K, \text{ for each } t \in T \tag{22}$$

Constraint (20) assures that the model cannot locate a large facility at a node l in time period t, and then in some later period after t, locate a small facility at that same node[1]. Similarly, this constraint also prevents solutions which would attempt to locate a large facility at a node l in some time period t after having previously located a more costly expanded facility at this same node in a previous time period. Constraint (21) again limits the number of facility types that the firm can locate at a node l in time period t to a maximum of one. Finally, constraint (22) precludes the model from selecting a small facility at a node l in time period t, and then selecting a large facility at this same node in period $t + 1$. In combination with (20) and (21), this assures that if Firm 3 locates a small facility at a node l, and then later plans to expand its capacity at node l, it must select a facility type $\overset{E}{y}{}_l^t$. As noted, the allocated period fixed location costs of this facility type reflect the total fixed costs associated with building a plant in two stages rather than in one continuous project. Note further that the notation of the original dynamic Stackelberg location model (9)-(16) requires minor modifications to substitute (20)-(22) for (15).

Table 12 displays the location costs and capacities for Firm 3 in model 5. In this example, Firm 3 can build either a small plant or a large plant for \$20 million and \$26 million, respectively. Alternatively, the firm can first build a small plant, and then for an additional \$7 million, expand its small plant into a

[1] Note that if the value of the F's are stated in their “present value” rather than their “nominal value”, one technically should multiply the left hand side of equation (20) by the quantity $(1 + r)$

	Small Plant			Large Plant			Expanded Large Plant		
Node	Q_l^{xt} (000)	TFLC[a]	PLC[b]	Q_l^{xt} (000)	TFLC	PLC	Q_l^{xt} (000)	TFLC	PLC
1	700	20	2.0	1,000	26	2.6	1,000	27	2.7
2	700	20	2.0	1,000	26	2.6	1,000	27	2.7
3	0	-	-	0	-	-	0	-	-
4	0	-	-	0	-	-	0	-	-
$\bar{Q}^t = 1,000,000$ units									

[a] TFLC = Total Fixed Location Cost $ (millions)
[b] PLC = Period Location Cost $ (millions)

Table 12: Location Costs and Capacity for Model 5

		Build Large Facility Immediately		Build Small Facility, Expand in Period 3	
Period	Production (000)	Capacity (000)	PLC[a] (F_l^t) $ (millions)	Capacity (000)	PLC (F_l^t) $ (millions)
1	515	1000	2.6	700	2.0
2	635	1000	2.6	700	2.0
3	819	1000	2.6	1000	2.7
Total Profits $ (millions)		40.4		41.6	

[a] PLC = Period Location Cost

Table 13: SNC Equilibrium Profits for Firm 3 When It Locates at Node 2 Using Model 5 (Staged Capacity Expansion with Increasing Demand Each Period)

large plant. The construction cost of this expanded, large plant thus totals $27 million.

Table 13 show the three period total profits for Firm 3 when it (1) locates a small plant at node 2 in period 1, and then expands the plant in period 3; or (2) locates a large plant in period 1. In this example, it turns out that a small plant can accommodate Firm 3's optimal production levels in periods 1 and 2, and that only in period 3 does Firm 3 require a larger plant to maximize its profits. This allows Firm 3 to maximize it profits over the planning horizon by using a "staged" capacity expansion approach rather than immediately constructing a large facility which will have significant unused capacity for several periods. The ability to evaluate a staged construction approach represents an additional advantage offered by a dynamic location model.

	Model 1	Model 2	Model 3	Model 4	Model 5
Node 1		✓		✓	✓
Node 2	✓		✓		

✓ : Node selected

Table 14: Firm 3's Location Decision Under Alternative Models

6 Conclusions

In this paper, we have developed a series of related static and dynamic models and numerical examples which illustrate the potential decision support capabilities of reaction function based facility location models. We observed in the small numerical examples the intuitive result that as Firm 3's information and ability to predict the market increased, so did its capability to maximize the profitability of its location decision. In particular, in model 3 where it could predict the reactions of Firm 1 and 2, Firm 3 generated more profitable solutions than in models 1 and 2. Models 4 and 5 illustrated the enhanced decision support created by employing a reaction function based approach in dynamic models. We observed that Firm 3's optimal location decision in model 4 differed from that in model 3, once the firm incorporated information about changing market demand over time. Finally, in model 5 when the firm could consider a staged construction approach, it generated an even more profitable solution than in model 4. In conclusion thus, we believe that a dynamic reaction function based modeling approach offers the potential to enhance the decision support capabilities yielded by plant and warehouse facility location models. We close this paper with Table 14 which depicts how Firm 3's location decision changed from model to model.

References

[1] Dafermos, S. C. (1988). "Sensitivity Analysis in Variatoinal Inequalities". *Mathematics of Operations Research*, 13:421-434.

[2] Dafermos, S. C., and Nagurney, A. (1987). "Oligopolistic and Competitive Behavior of Spatially Separated Markets". *Regional Science and Urban Economics*, 17:245-254.

[3] Dockner, E. (1992). "A Dynamic Theory of Conjectural Variations". *Journal of Industrial Economics XL*, no. 4:377-395

[4] Friedman, J. W. (1977). *Oligopoly and the Theory of Games.* Amsterdam: North-Holland.

[5] Gabay, D. and Moulin, H. (1980). "On the Uniqueness and Stability of Nash-Equilibria in Noncooperative Games." in *Applied Stochastic Control*

in Econometrics and Management Sciences, eds. A. Bensoussan, P. Kleindorfer and C.S. Tapeiro. Amsterdam: North-Holland.

[6] Harker, P. T. (1984). "A Variational Inequality Approach for the Determination of Oligopolistic Market Equilibrium". *Mathematical Programming*, 30:105-111.

[7] Harker, P. T. (1986). "Alternative Models of Spatial Competition". *Operations Research*, 34:410-425.

[8] Haurie, A. and Marcotte, P. (1985). "On the Relationship between Cournot-Nash and Wardrop Equilibria". *Networks*, 15:295-308.

[9] Kyparisis, J. (1987). "Sensitivity Analysis Framework for Variational Inequalities", *Mathematical Programming*, 38:203-213

[10] Kyparisis, J. (1989). "Solution Differentiability for Variational Inequalities and Nonlinear Programming Problems". Miami, FL: Floridal International University, Department of Decision Sciences and Information Systems, College of Business Administration, Working Paper.

[11] Lions, J. L. and Stampacchia, G. (1967). "Variational Inequalities". *Communications on Pure and Applied Mathematics*. 20:493-519.

[12] Marcotte, P. (1987). "Algorithms for the Network Oligopoly Problem". *Journal of Operational Research Society* 38, no. 11:1051-1065.

[13] Miller, T. C., Friesz, T. L. and Tobin, R. L. (1995). *Equilibrium Facility Location on Networks*. Springer-Verlag (Heidelberg).

[14] Miller, T. C., Tobin, R. L. and Friesz, T. L. (1991). "Stackelberg Games on a Network with Cournot-Nash Oligopolistic Competitors". *Journal of Regional Sciences* 31:435-454.

[15] Miller, T. C., Tobin, R. L. and Friesz, T. L. (1992). "Network Facility Location Models in Stackelberg-Nash-Cournot Spatial Competition". *Papers in Regional Sciences: The Journal of the RSAI* 71, no.3: 277-291.

[16] Miyagi, T. (1991). "A Computational Procedure for Determination of Oligopolistic Spatial Price Equilibrium", *Papers in Regional Sciences* 70, no.2:185-200.

[17] Murphy, F. H., Sherali, H. D., and Soyster, A. L. (1982). "A Mathematical Programming Approach for Determining Oligopolistic Market Equilibrium", *Mathematical Programming* 24:92-106.

[18] Nagurney, A. (1988). "Algorithms for Oligopolistic Market Equilibrium Problems". *Regional Science and Urban Economics*. 18:425-445.

[19] Nagurney, A., Dupuis, P., and Zhang, D. (1994). "A Dynamical Systems Approach for Network Oligopolies and Variational Inequalities'. *Annals of Regional Science* 28:263-283.

[20] Pang, J.-S. (1988). "Solution Differentiability and Continuation of Newton's Method for Variational Inequalities Problems over Polyhedral Sets. Baltimore, MD: The Johns Hopkins University, Department of Mathematical Sciences. Working Paper.

[21] Qui, Y. and Magnanti, T. L. (1989). "Sensitivity Analysis for Variational Inequalities Defined on Polyhedral Sets". *Mathematics of Operations Research* 14(3):410-432.

[22] Tobin, R. L. (1986). "Sensitivity Analysis for Variational Inequalities". *Journal of Optimization Theory and Applications*. 48:191-204.

[23] Tobin, R. L., Miller, T. C. and Friesz, T. L. (1995). "Incorporating Competitors' Reactions in Facility Location Decisions: A Market Equilibrium Approach". *Location Science* 3(4):239-253.

[24] Wie, B. and Tobin, R. L. (1997). "A Dynamical Spatial Cournot-Nash Equilibrium Model and an Algorithm". *Computational Economics* 10:15-45.

Appendix A Coefficient For Demand, Production and Transportation Cost Functions, and Firm 3's Location Cost Data

Period	Node $l \in K$	All Firms : Models 1 to 3		All Firms : Models 4 to 5		Firm 1	Firm 2	Firm 3
		α_l^t	B_l^t	α_l^t	B_l^t	$c1$	$c2$	$c3$
1	1	72,000	43	72,000	43	58	53	47
	2	28,000	16	28,000	16	80	75	44
	3	25,000	7	25,000	7	84	80	-
	4	22,000	11	22,000	11	78	72	-
2	1	72,000	43	82,000	43	58	53	47
	2	28,000	16	48,000	16	80	75	44
	3	25,000	7	35,000	7	84	80	-
	4	22,000	11	32,000	11	78	72	-
3	1	72,000	43	94,000	43	58	53	47
	2	28,000	16	60,000	8	80	75	44
	3	25,000	7	40,000	7	84	80	-
	4	22,000	11	34,000	8	78	72	-

Table A-1: Demand and Production Coefficients

Arc t_{jl}^t	Firm 1	Firm 2	Firm 3
t_{11}	1.2	1.5	1.0
t_{12}	5.3	5.3	5.3
t_{13}	5.9	5.8	5.7
t_{14}	5.5	5.6	5.7
t_{21}	5.4	5.3	5.2
t_{22}	1.4	1.3	1.3
t_{23}	6.0	5.9	5.8
t_{24}	6.3	6.2	6.0
t_{31}	4.5	4.4	4.5
t_{32}	4.3	4.2	4.3
t_{33}	1.1	1.3	1.2
t_{34}	4.0	4.2	4.1
t_{41}	4.4	4.3	4.1
t_{42}	4.0	4.1	4.2
t_{43}	3.9	4.0	3.8
t_{44}	1.3	1.5	1.2

Table A-2: Transportation Coefficients (For All Periods)

Node	Q_l^{xt} (000)	Total Fixed Loc. Cost \$ (millions)	Period Loc. Cost \$ (millions)
1	1,000	26	2.6
2	1,000	26	2.6
3	0	-	-
4	0	-	-
$\bar{Q}^t = 1,000,000$ units			

Table A-3: Location Costs and Capacity for Models 1 through 4