Risk-Averse Network Design with Behavioral Conditional Value-at-Risk for Hazardous Materials Transportation

Liu Su and Changhyun Kwon*

Department of Industrial and Management Systems Engineering, University of South Florida

April 10, 2018

Abstract

We consider a road-ban problem in hazardous materials (hazmat) transportation. We formulate the problem as a network design problem to select a set of closed road segments for hazmat traffic and obtain a bi-level optimization problem. While modeling probabilistic route-choices of hazmat carriers by the random utility model (RUM) in the lower level, we consider a risk-averse measure called conditional value-at-risk (CVaR) in the upper level, instead of the widely used expected risk measure. Using RUM and CVaR, we quantify the risk of having hazmat accidents and large consequences, and design the network policy for road-bans accordingly. While CVaR has been used in hazmat routing problems, this paper is the first attempt to apply CVaR in risk averse hazmat network design problems considering stochastic route-choices of hazmat carriers. The resulting problem is a mixed integer nonlinear programming problem, for which we devise a line search approach combined with Benders decomposition. We demonstrate the efficiency of the proposed computational method with case studies. By applying CVaR to the route-choice behavior of hazmat carriers, we protect the road network from undesirable route-choices that may lead to severe consequences.

Keywords: transportation; hazardous materials; network design; conditional value-at-risk; Benders decomposition

1 Introduction

Hazardous materials (hazmat) are defined as materials that can pose an unreasonable threat to the public and the environment (Federal Motor Carrier Safety Administration, 2016b) and about 1 million shipments of hazardous materials crisscross the United States every day. In the past decade, there have been 166,968 incidents causing 105 fatalities, resulting in 2,129 injuries and costing $820,432,788 of damages (Pipeline and Hazardous Materials Safety Administration, 2016). Widely

*Corresponding Author: `chkwon@usf.edu`

used for hazardous materials transportation are cargo tank trucks. Improving the truck safety on national highways and reducing the transportation incidents are significantly important for the public and the environment.

On account of the large number of hazardous materials transported via roads, the government and transportation agencies pay attention to reduce the risk of potential catastrophic accidents by hazmat. There are various policies and tools in both truck operations and network designs for mitigating the risk. The Federal Motor Carrier Safety Administration (FMCSA), in association with the Pipeline and Hazardous Materials Safety Administration (PHMSA) and industry partners, created training tools for commercial hazmat carriers of cargo tank motor vehicles transporting hazardous materials, since 78% of rollovers involve the hazmat carrier's error (Federal Motor Carrier Safety Administration, 2016a). Besides, the government and agencies provide the road-ban policies to protect the public and the environment from severe accident consequences of hazmat. In a road network design for hazmat transportation, the government can close certain road segments for hazmat traffic while hazmat carriers can determine routes to transport hazmat without using closed roads.

Typically, hazmat network design problems are formulated as bi-level optimization problems (Kara and Verter, 2004; Erkut and Gzara, 2008; Gzara, 2013; Fontaine and Minner, 2018; Sun et al., 2016b, 2018). The upper level *selects* a set of road segments to be closed aiming at minimizing the risk of hazmat transportation in the network. The lower level *predicts* the carriers' routes to transport hazmat from origin-destination (OD) pairs. Modeling and predicting route choices of carriers is essential to determine the risk associated with a hazmat transportation network. Most studies on hazmat network design utilize the shortest path problem to model the route choices, although routing behavior is uncertain. An exception is Sun et al. (2018), who assumes hazmat carriers are satisficing decision makers; hence route choices of hazmat carriers can be suboptimal and uncertain.

To consider the uncertainty of route choices, random utility model (RUM), random regret-minimization (RRM), bounded rationality (BR), cumulative prospect theory (CPT), fuzzy logic model (FLM) and dynamic learning models (DLM) are developed (Sun et al., 2016a). Among these methods, RUM models the uncertainty of routing with probabilistic-route choices. McFadden (1975) first proposed RUM to model the choice behavior. In RUM, it is assumed that users' utility depends on both a fixed effect and a random observation error. Daganzo and Sheffi (1977) assumed that observation errors are normally distributed ending up with Multinomial Probit (MNP) model. MNP is lack of tractability for researchers to perform further analysis, because it cannot provide an explicit formula which relates choice probabilities and known factors. Later, Sheffi (1985) proposed Multinomial Logit (MNL) model by assuming that observation errors are from Gumbel distribution. Other logit-type models (Cascetta et al., 1996; Ben-Akiva and Bierlaire, 1999; Ramming, 2001) cannot be applied in network design problems, since they need to evaluate path set beforehand. The simple explicit form of MNL to describe users' stochastic behavior makes it incorporable with further analysis. By using MNL in transportation, the route choice probabilities can directly relate

Table 1: Have *risk-averse* approaches been used in hazmat transportation problems? RO represents robust optimization.

Paper	Source of Uncertainty			Context
	Route-Choice	Accident Consequence	Data	
Toumazis et al. (2013)	-	VaR/CVaR	-	Routing
Toumazis and Kwon (2016)	-	CVaR	RO	Routing
Sun et al. (2016b)	-	-	RO	Network Design
Sun et al. (2018)	RO	-	-	Network Design
This Paper	CVaR	CVaR	-	Network Design

to route costs.

Most hazmat transportation network designs only address the economical perspective or consider simple risk measures such as expectation of accident consequences. In risk management, value-at-risk (VaR), also known as α-quantile, once was commonly used to measure risk ignoring the left tail of loss distribution. Its lack of subadditivity and convexity, as discussed by Artzner et al. (1997, 1999), however, leads researchers' attention to a coherent measure: conditional value-at-risk (CVaR). While both VaR (Duffie and Pan, 1997) and CVaR (Rockafellar and Uryasev, 2000) have been applied to financial portfolio optimization problems, they have also been applied to hazmat routing. Toumazis et al. (2013) proposed VaR and CVaR minimization for hazmat routing. Toumazis and Kwon (2013) focused on using CVaR to route on time-dependent networks. Considering the uncertainty of accident data, Toumazis and Kwon (2016) proposed a worst CVaR minimization problem for hazmat routing.

Our main contribution is that we introduce a *risk-averse* CVaR measure to both *probabilistic behavior* of hazmat carriers and *probabilistic consequences* from hazmat accidents in hazmat network design problems. To the best of our knowledge, this paper is the first attempt to mitigate both factors via averse risk measures. Sun et al. (2018) considered the worst-case behavior of hazmat carriers using the notion of bounded rationality to derive a robust network design, while the expected risk (ER) is used to measure the risk from hazmat accidents. Their worst-case approach is similar to robust optimization methods without assuming any probability distribution for route choices. In contrast, we consider CVaR for both probabilistic factors. On the other hand, unlike Toumazis and Kwon (2016) who considered the data uncertainty, we assume that hazmat accident probabilities and consequences at each road segment are available and deterministic. Table 1 highlights our main contribution and shows the differences between our work and other available approaches in the literature. We analyze the proposed CVaR minimization problem for hazmat network design theoretically and develop an efficient algorithm that combines line search with Benders decomposition to solve the problem. In addition, we provide case studies on realistic road networks to confirm the validity of CVaR concept incorporating probabilistic-route choices and the practicability of the proposed algorithms.

2 A Deterministic Model for Network Design

In this section, we review a deterministic model for hazmat transportation network design. Later, we extend the deterministic model to consider CVaR and uncertain route choices.

Let us consider a transportation network $G = (\mathcal{N}, \mathcal{A})$ where $\mathcal{N}$ is the set of nodes and $\mathcal{A}$ is the set of arcs. In a multi-commodity transportation network, let $\mathcal{S}$ denote the set of shipments. In practice, S specifies the OD pair, and the kind of hazmat. Let N^s be the demand of shipment $s \in \mathcal{S}$ that represents the number of trucks carrying hazmat. Each arc (i, j) is known with the travel cost t_{ij}, the accident probability p_{ij}, and the accident consequence c^s_{ij} for each shipment $s \in \mathcal{S}$. Accidents caused by various kinds of hazmat can have different influences on a road network making it possible that different shipments can have different accident consequences. Let $\mathcal{K}_s$ be the set of available paths for shipment $s \in \mathcal{S}$. To transport shipment $s \in \mathcal{S}$, the approximated risk distribution for a single demand (truck) along path $k \in \mathcal{K}_s$ can be written as follows (Jin and Batta, 1997):

$$\Pr\{R^{sk} = x\} \approx \begin{cases} 1 - \displaystyle\sum_{(i,j)\in\mathcal{A}^k} p_{ij} & \text{if } x = 0 \\ p_{ij} & \text{if } x = c^s_{ij} \text{ for some } (i,j) \in \mathcal{A}^k \end{cases} \tag{1}$$

where $\mathcal{A}^k$ is the set of arcs for path k. One of the most common approaches that regulators use to measure the risk is expected value of consequences for potential hazmat truck accidents. It is a common assumption that hazmat carriers travel along the shortest path. We also assume that hazmat carriers only follow the shortest path in the deterministic model for hazmat transportation network design. Erkut and Gzara (2008) solved a bi-level hazmat transport network design problem based on an arc-based formulation. Verter and Kara (2008) proposed a path-based approach for hazmat transport network design by simplifying the shortest path problem with the closet assignment constraints. Similarly, the deterministic path-based hazmat transportation network design is formulated as follows:

$$\min_{y,z} \sum_{s\in\mathcal{S}} \sum_{k\in\mathcal{K}_s} \sum_{(i,j)\in\mathcal{A}} N^s p_{ij} \delta^{sk}_{ij} c^s_{ij} \gamma^{sk} \tag{2}$$

$$\text{s.t. } z^{sk} \geq \sum_{(i,j)\in\mathcal{A}} \delta^{sk}_{ij} y_{ij} - \sum_{(i,j)\in\mathcal{A}} \delta^{sk}_{ij} + 1, \qquad \forall s \in \mathcal{S}, \forall k \in \mathcal{K}_s \tag{3}$$

$$z^{sk} \leq y_{ij} - \delta^{sk}_{ij} + 1, \qquad \forall s \in \mathcal{S}, \forall k \in \mathcal{K}_s, \forall (i,j) \in \mathcal{A} \tag{4}$$

$$\sum_{k\in\mathcal{K}_s} z^{sk} \geq 1, \qquad \forall s \in \mathcal{S} \tag{5}$$

$$\gamma^{sk} \leq z^{sk}, \qquad \forall s \in \mathcal{S}, k \in \mathcal{K}_s \tag{6}$$

$$\gamma^{sk} \geq z^{sk} - \sum_{j=1}^{k-1} z^{sj}, \qquad \forall s \in \mathcal{S}, k \in \mathcal{K}_s \tag{7}$$

$$\sum_{(i,j)\in\mathcal{A}} (1 - y_{ij}) \leq N \tag{8}$$

$$\gamma^{sk}, z^{sk} \text{ binary}, \qquad \forall s \in \mathcal{S}, \forall k \in \mathcal{K}_s \tag{9}$$

$$y_{ij} \text{ binary}, \qquad \forall (i,j) \in \mathcal{A} \tag{10}$$

where y is the design variable, z is the path availability variable and γ is the route-choice variable. If arc (i, j) is open for transportation of hazmat, $y_{ij} = 1$; otherwise, $y_{ij} = 0$. If path k is available for transportation of shipment $s \in \mathcal{S}$, $z^{sk} = 1$; otherwise, $z^{sk} = 0$. If path k is chosen for transportation of shipment $s \in \mathcal{S}$, $\gamma^{sk} = 1$; otherwise, $\gamma^{sk} = 0$. In addition, δ_{ij}^{sk} is the parameter to define a path. If $\delta_{ij}^{sk} = 1$, arc (i, j) is on path k for shipment s; otherwise, $\delta_{ij}^{sk} = 0$. In the single-level problem by a path-based formulation, the objective minimizes the ER as (2) shows. Path-based network design constraints are defined by (3)–(10). Constraints (3) and (4) define path availability for shipments. A path is available only when all arcs on the path are open. If there exist closed arcs on a path, the path is out of service. In addition, at least one path for a shipment is available to ensure transportation as (5) shows. Constraints (6) state that the chosen path for shipments must come from available paths. All paths for a shipment are sorted from 1 to k by lengths meaning that the length of path 1 for any shipment has shortest length among all possible paths. Constraints (7) guarantee that the available path with the smallest index is used for each shipment. Because of the sorted path data, (7) is equivalent to the shortest path problem in a path-based context. Due to the cost associated with closing arcs, (8) restricts the number of closed arcs. Constraints (9) and (10) are binaries for decision variables. The path-based hazmat transportation network design problem is a mixed-integer linear programming (MILP) problem.

3 Hazmat Risk Modeling with Probabilistic Route Choices

Traditionally, researchers model the risk distribution for a hazmat transportation network by assuming that hazmat carriers choose the shortest path. To consider the uncertainty of route choices, probabilistic route choice models are used. In probabilistic route choice models, hazmat carriers choose an available path with a probability. The risk distribution for a hazmat transportation network is redefined to incorporate with probabilistic route choices. In this section, probabilistic route choice models are reviewed and utilized in risk distribution for hazmat transportation network.

3.1 Random Utility and Probabilistic Route Choice Models

RUM assumes that the utility of a choice that decision makers perceive comes from two sources: a deterministic (observable) component and a random (unobservable) component. In the context of route choices, the utility U^{sk} of path k for shipment $s \in \mathcal{S}$ is defined by:

$$U^{sk} = -\theta^s t^{sk} + \xi^{sk} \tag{11}$$

where t^{sk} is the generalized cost of observable attributes, θ^s is a positive parameter and ξ^{sk} is a random variable. Usually, t^{sk} is the travel time. It is assumed to be additive with respect to arc costs.

$$t^{sk} = \sum_{(i,j)\in\mathcal{A}} t_{ij}\delta_{ij}^{sk} \tag{12}$$

where t_{ij} is the generalized travel cost associated with arc (i,j), and $\delta_{ij}^{sk} = 1$ if arc (i,j) is on path k for shipment $s \in \mathcal{S}$ and 0 otherwise.

Different distributions for random components ξ^{sk} result in various forms of probabilities π^{sk} of choosing path $k \in \mathcal{K}_s$ for shipment $s \in \mathcal{S}$. By assuming that the random component ξ^{sk} are independently and identically from Gumbel distribution, MNL model can be obtained as follows:

$$\pi^{sk} = \frac{\rho^{sk}}{\sum_{l\in\mathcal{K}_s}\rho^{sl}} \tag{13}$$

$$\rho^{sk} = e^{-\theta^s t^{sk}} \tag{14}$$

for all $s \in \mathcal{S}, k \in \mathcal{K}_s$.

There exist other logit-type models with different formulations of ρ^{sk} (Prashker and Bekhor, 2004). In C-logit model, a commonality factor is introduced while a path size is defined in path-size logit model. Both the commonality factor and the path size are used to measure the similarity among paths and address some overlapping problems which MNL cannot capture. To obtain the commonality factor and the path size, however, we need to know the path set $\mathcal{K}_s$ for shipment $s \in \mathcal{S}$ beforehand. Therefore, C-logit model and path-size model cannot apply to the network design problem. The simple form of MNL is used to model the uncertain route choices.

3.2 The Risk Distribution for Hazmat Transportation

In this section, the risk distribution for hazmat transportation is defined incorporating with the probabilistic route choice model. Various shipments $s \in \mathcal{S}$ can have different accident consequences. Let $\mathcal{A}^k$ denote the set of arcs for path $k \in \mathcal{K}_s$ to transport shipment $s \in \mathcal{S}$. It is assumed that hazmat carriers operate independently. Among N^s demands of hazmat for shipment $s \in \mathcal{S}$, demand (truck) 1 and demand (truck) 2 have the same risk distribution along path $k \in \mathcal{K}_s$. Choosing path $k \in \mathcal{K}_s$ to transport shipment $s \in \mathcal{S}$, the risk distribution for n-th truck can be approximated as follows (Jin and Batta, 1997):

$$R_n^{sk} = \begin{cases} 0 & \text{with probability } 1 - \sum_{(i,j)\in\mathcal{A}^k} p_{ij} \\ c_{ij}^s & \text{with probability } p_{ij} \text{ for } (i,j) \in \mathcal{A}^k \end{cases} \tag{15}$$

When there are multiple paths available for each truck to transport shipment $s \in \mathcal{S}$, we assume that a path is chosen at the probability described by the probabilistic route choice model introduced in Section 3.1. Let R_n^s be the random risk variable for n-th truck to transport $s \in \mathcal{S}$, distributed among all available paths in $\mathcal{K}_s$. Under the consideration of available paths, the probability of taking risk x of shipment $s \in \mathcal{S}$ by n-th truck is:

$$\Pr\left[R_n^s = x\right] = \sum_{k \in \mathcal{K}_s} \Pr\left[R_n^s = x \mid \text{path } k \text{ chosen}\right] \Pr\left[\text{path } k \text{ chosen for shipment } s\right] \tag{16}$$

$$= \sum_{k \in \mathcal{K}_s} \Pr\left[R_n^{sk} = x\right] \pi^{sk} \tag{17}$$

where π^{sk} is given in (13). Hence, R_n^s is distributed as follows:

$$R_n^s = \begin{cases} 0 & \text{with probability } 1 - \sum_{k \in \mathcal{K}_s} \sum_{(i,j) \in \mathcal{A}^k} \pi^{sk} p_{ij} \\ c_{ij}^s & \text{with probability } p_{ij} \sum_{k \in \mathcal{K}_s} \pi^{sk} \delta_{ij}^{sk} \text{ for } (i,j) \in \bigcup_{k \in \mathcal{K}_s} \mathcal{A}^k \end{cases} \tag{18}$$

where δ_{ij}^{sk} is the incidence parameter for $s \in \mathcal{S}, k \in \mathcal{K}_s, (i,j) \in \mathcal{A}$. If $\delta_{ij}^{sk} = 1$, arc (i,j) is on path k for shipment s; if $\delta_{ij}^{sk} = 0$, arc (i,j) is not on path k for shipment s. The risk for a given transportation network comes from all demands among all shipments. Therefore, the risk for a transportation network is:

$$R = \sum_{s \in \mathcal{S}} \sum_{n=1}^{N^s} R_n^s \tag{19}$$

Since different trucks are operated separately transporting multiple units of hazmat, we can assume that the risks for multiple units of hazmat among all shipments are independently distributed. According to the North America data on hazmat transportation accident statistics, the probabilities of an accident to take place are very small ranging from 10^{-8} to 10^{-6} (Abkowitz et al., 1992). Utilizing

$$p_{ij} p_{i'j'} \approx 0 \tag{20}$$

for all $(i,j), (i',j') \in \mathcal{A}$, we can obtain the probability that the risk variable becomes 0 as follows:

$$\begin{aligned} \Pr\left[R = 0\right] &= \prod_{s \in \mathcal{S}} \prod_{n=1}^{N^s} \Pr\left[R_n^s = 0\right] \\ &= \prod_{s \in \mathcal{S}} \prod_{n=1}^{N^s} \left(1 - \sum_{k \in \mathcal{K}_s} \sum_{(i,j) \in \mathcal{A}^k} \pi^{sk} p_{ij}\right) \\ &\approx \prod_{s \in \mathcal{S}} \left(1 - N^s \sum_{k \in \mathcal{K}_s} \sum_{(i,j) \in \mathcal{A}^k} \pi^{sk} p_{ij}\right) \\ &= 1 - \sum_{s \in \mathcal{S}} \sum_{k \in \mathcal{K}_s} \sum_{(i,j) \in \mathcal{A}^k} N^s \pi^{sk} p_{ij} \end{aligned} \tag{21}$$

and for each $c_{ij}^s : s \in \mathcal{S}, (i,j) \in \mathcal{A}$:

$$\begin{aligned}
\Pr\left[R = c_{ij}^s\right] &= \Pr\left[\sum_{s\in\mathcal{S}}\sum_{n=1}^{N^s} R_n^s = c_{ij}^s\right] \\
&\approx \sum_{n=1}^{N^s} \Pr\left[R_n^s = c_{ij}^s\right] \\
&= \sum_{n=1}^{N^s} p_{ij} \sum_{k\in\mathcal{K}_s} \pi^{sk}\delta_{ij}^{sk} \qquad (22) \\
&= \sum_{k\in\mathcal{K}_s} N^s \pi^{sk} p_{ij} \delta_{ij}^{sk} \qquad (23)
\end{aligned}$$

Therefore, the approximated risk distribution for hazmat transportation network is

$$R = \begin{cases} 0 & \text{with probability } 1 - \sum_{s\in\mathcal{S}}\sum_{k\in\mathcal{K}_s}\sum_{(i,j)\in\mathcal{A}^k} N^s\pi^{sk}p_{ij} \\ c_{ij}^s & \text{with probability } \sum_{k\in\mathcal{K}_s} N^s\pi^{sk}p_{ij}\delta_{ij}^{sk} \text{ for } (i,j)\in\mathcal{A}, s\in\mathcal{S}. \end{cases} \qquad (24)$$

4 The CVaR Minimization Model for Network Design

In this section, a CVaR minimization network design model considering drivers' probabilistic route choices is proposed. It is well-known that CVaR is a general, coherent and risk-averse measure (Rockafellar and Uryasev, 2002). For any random loss X, the VaR and CVaR are introduced in Definitions 1 and 2, respectively. CVaR can also be redefined as an optimization problem as Theorem 1 shows.

Definition 1 (VaR Measure)**.** *The value-at-risk (VaR) is defined as follows:*

$$\mathsf{VaR}_p(X) = \inf\{x : \Pr[X \leq x] \geq p\} \qquad (25)$$

where $p \in (0,1)$ is a threshold probability.

Definition 2 (CVaR Measure)**.** *The conditional value-at-risk (CVaR) is defined as follows:*

$$\mathsf{CVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathsf{VaR}_p(X)\,\mathrm{d}p \qquad (26)$$

for a threshold probability $\alpha \in (0,1)$ where $\mathsf{VaR}_p(X)$ is shown in Definition 1.

Theorem 1 (Rockafellar and Uryasev, 2002)**.** *For $r \in \mathbb{R}$, let us define*

$$\Phi_\alpha(r;X) = r + \frac{1}{1-\alpha}\mathbb{E}[X-r]^+,$$

where $[x]^+ = \max(x, 0)$*. Then the CVaR measure is equivalent to:*

$$\mathsf{CVaR}_\alpha(X) = \min_{r \in \mathbb{R}} \Phi_\alpha(r; X) \tag{27}$$

4.1 Route-Choice Probabilities Depending on Network Design

To introduce the CVaR measure for hazmat transportation, the route-choice probabilities depending on network design are clarified. Let y be the path-based network design variables and z be the path availability variables here. If arc (i, j) is open for transportation of hazmat, $y_{ij} = 1$; otherwise, $y_{ij} = 0$. If path k is available for transportation of shipment $s \in \mathcal{S}$, $z^{sk} = 1$; otherwise, $z^{sk} = 0$. The route-choice probabilities are formulated as follows:

$$z^{sk} \geq \sum_{(i,j)\in\mathcal{A}} \delta_{ij}^{sk} y_{ij} - \sum_{(i,j)\in\mathcal{A}} \delta_{ij}^{sk} + 1, \qquad \forall s \in \mathcal{S}, \forall k \in \mathcal{K}_s \tag{28}$$

$$z^{sk} \leq y_{ij} - \delta_{ij}^{sk} + 1, \qquad \forall s \in \mathcal{S}, \forall k \in \mathcal{K}_s, \forall (i,j) \in \mathcal{A} \tag{29}$$

$$\sum_{k\in\mathcal{K}_s} z^{sk} \geq 1, \qquad \forall s \in \mathcal{S} \tag{30}$$

$$\sum_{(i,j)\in\mathcal{A}} (1 - y_{ij}) \leq N \tag{31}$$

$$\pi^{sk} = \frac{\rho^{sk} z^{sk}}{\sum_{l\in\mathcal{K}_s} \rho^{sl} z^{sl}}, \qquad \forall s \in \mathcal{S}, \forall k \in \mathcal{K}_s \tag{32}$$

$$z^{sk} \text{ binary }, \qquad \forall s \in \mathcal{S}, \forall k \in \mathcal{K}_s \tag{33}$$

$$y_{ij} \text{ binary }, \qquad \forall (i,j) \in \mathcal{A} \tag{34}$$

Equations (28) and (29) determine the path availabilities, similarly as in Section 2. Equation (30) constrains that there exists at least one path for shipment $s \in \mathcal{S}$. Equation (31) states that at most N arcs can be closed in the network.

Hazmat carriers, however, do not necessarily choose the shortest path in all cases. To model the uncertainty of driver behaviors, probabilistic route choice model is introduced. In the proposed model, we assume that carriers choose paths among all available paths by estimating their utilities. Then, we use RUM to model carriers' probabilistic behavior and MNL to further simply the stochastic route-choice. Equation (32) shows the route-choice probabilities among all available paths. If path $k \in \mathcal{K}_s$ is unavailable for shipment $s \in \mathcal{S}$, namely $z^{sk} = 0$, its route-choice probability is 0; otherwise, the route-choice probability can be given by (13) and (14).

4.2 The CVaR Minimization Model for Network Design

This section shows the CVaR minimization model for hazmat network design. The distribution of risk introduced in Section 3.2 and the route-choice probabilities in Section 4.1 can model the CVaR minimization network design problem. Let

$$\Phi_\alpha(r;\pi) = r + \frac{1}{1-\alpha}\mathbb{E}\left[R-r\right]^+ \tag{35}$$

$$\approx r + \frac{1}{1-\alpha}\Bigg\{\Bigg(1-\sum_{s\in\mathcal{S}}\sum_{k\in\mathcal{K}_s}\sum_{(i,j)\in\mathcal{A}^k} N^s\pi^{sk}p_{ij}\Bigg)\left[0-r\right]^+$$

$$+\sum_{(i,j)\in\mathcal{A}}\sum_{s\in\mathcal{S}}\sum_{k\in\mathcal{K}_s} N^s\pi^{sk}p_{ij}\delta_{ij}^{sk}\left[c_{ij}^s-r\right]^+\Bigg\} \tag{36}$$

$$\approx r + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}}\sum_{s\in\mathcal{S}}\sum_{k\in\mathcal{K}_s} N^s\pi^{sk}p_{ij}\delta_{ij}^{sk}\left[c_{ij}^s-r\right]^+ \tag{37}$$

We use the optimization of CVaR in Theorem 1 to define the CVaR measure in hazmat transportation network,

$$\mathsf{CVaR}_\alpha = \min_{r\in\mathbb{R}^+}\Phi_\alpha(r;\pi) \approx \min_{r\in\mathbb{R}^+}\left[r + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}}\sum_{s\in\mathcal{S}}\sum_{k\in\mathcal{K}_s} N^s\pi^{sk}p_{ij}\delta_{ij}^{sk}\left[c_{ij}^s-r\right]^+\right]. \tag{38}$$

Therefore, the CVaR minimization model is,

$$\min_{\pi\in\Omega}\mathsf{CVaR}_\alpha = \min_{\pi\in\Omega, r\in\mathbb{R}^+}\Phi_\alpha(r;\pi) \tag{39}$$

$$\approx \min_{\pi\in\Omega, r\in\mathbb{R}^+}\left[r + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}}\sum_{s\in\mathcal{S}}\sum_{k\in\mathcal{K}_s} N^s\pi^{sk}p_{ij}\delta_{ij}^{sk}\left[c_{ij}^s-r\right]^+\right] \tag{40}$$

where Ω can be defined by

$$\Omega = \{\pi : \exists y, z \text{ such that (28)–(34) hold}\}. \tag{41}$$

4.3 The Model Analysis

The CVaR minimization model for hazmat transportation network design (40) is a mixed integer nonlinear programming problem. If a network is complicated with a large demand of shipments, it becomes very difficult to solve the problem. In the proposed model, variable r only has an impact on the objective function and does not exist in constraints. Because the objective function is linear with r within each interval between two consecutive c_{ij}^s values, the optimal r value lies in $\Theta = \{0\}\cup\{c_{ij}^s : \forall(i,j)\in\mathcal{A}, s\in\mathcal{S}\}$ (Toumazis et al., 2013). The CVaR minimization model (40) is reformulated as:

$$\min_{r\in\Theta} f_\alpha(r) \tag{42}$$

where

$$f_\alpha(r) = \min_{\pi\in\Omega}\Phi_\alpha(r;\pi).$$

Given a large network with various kinds of hazmat, set Θ becomes large. To obtain the optimal solution of the proposed model, we should solve a large number of $f_\alpha(r)$. If some r values can be eliminated without solving optimization problems, the computation can be more efficient. Analysis is conducted to explore which r values can be eliminated from being optimal solutions for the proposed model. Let

$$0 = r_0 \leq r_1 \leq r_2 \leq ... \leq r_{q-1} \leq r_q \leq r_{q+1} \leq ... \leq r_{M_\mathcal{A}} \tag{43}$$

where r_q is the q-th smallest value in $\{c^s_{ij} : \forall(i,j) \in \mathcal{A}, s \in \mathcal{S}\}$ and $M_\mathcal{A}$ is the number of unique values in $\{c^s_{ij} : \forall(i,j) \in \mathcal{A}, s \in \mathcal{S}\}$. For each $q = 0, 1, \cdots, M_\mathcal{A} - 1$, we have

$$\begin{aligned}
\Phi_\alpha(r_{q+1};\pi) - \Phi_\alpha(r_q;\pi) &= r_{q+1} + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}}\sum_{s\in\mathcal{S}}\sum_{k\in\mathcal{K}_s} N^s\pi^{sk}p_{ij}\delta^{sk}_{ij}\left[c^s_{ij} - r_{q+1}\right]^+ \\
&\quad - r_q - \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}}\sum_{s\in\mathcal{S}}\sum_{k\in\mathcal{K}_s} N^s\pi^{sk}p_{ij}\delta^{sk}_{ij}\left[c^s_{ij} - r_q\right]^+ \\
&= r_{q+1} - r_q \\
&\quad - \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}}\sum_{s\in\mathcal{S}}\sum_{\substack{k\in\mathcal{K}_s \\ c^s_{ij}\geq r_{q+1}}} N^s\pi^{sk}p_{ij}\delta^{sk}_{ij}(r_{q+1} - r_q) \\
&= (r_{q+1} - r_q)\left(1 - \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}}\sum_{s\in\mathcal{S}}\sum_{\substack{k\in\mathcal{K}_s \\ c^s_{ij}\geq r_{q+1}}} N^s\pi^{sk}p_{ij}\delta^{sk}_{ij}\right).
\end{aligned} \tag{44}$$

Theorem 2. *Consider an index $q \in \{0, 1, \ldots, M_\mathcal{A}\}$ such that the following condition holds:*

$$\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}}\sum_{s\in\mathcal{S}}\sum_{\substack{k\in\mathcal{K}_s \\ c^s_{ij}\geq r_{q+1}}} N^s p_{ij}\delta^{sk}_{ij} \leq 1 \tag{45}$$

Then we can show that

$$\Phi_\alpha(r_q;\pi) \leq \Phi_\alpha(r_{q+1};\pi) \tag{46}$$

for all $\pi \in \Omega$. Further

$$f_\alpha(r_q) \leq f_\alpha(r_{q+1}) \leq \cdots \leq f_\alpha(r_{M_\mathcal{A}}) \tag{47}$$

Proof of Theorem 2. Given condition (45), we have

$$\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}}\sum_{s\in\mathcal{S}}\sum_{\substack{k\in\mathcal{K}_s \\ c^s_{ij}\geq r_{q+1}}} N^s\pi^{sk}p_{ij}\delta^{sk}_{ij} \leq 1$$

for any π, since $\pi^{sk} \in [0,1]$ is the probability associated with path $k \in \mathcal{K}_s$ for shipment $s \in \mathcal{S}$. Based on (44), for any route-choice probabilities $\pi \in \Omega$

$$\Phi_\alpha(r_q;\pi) \leq \Phi_\alpha(r_{q+1};\pi). \tag{48}$$

Note that

$$\frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}}\sum_{s\in\mathcal{S}}\sum_{\substack{k\in\mathcal{K}_s\\ c_{ij}^s\geq r_{M_{\mathcal{A}}}}} N^s\pi^{sk}p_{ij}\delta_{ij}^{sk} \leq \cdots \leq \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}}\sum_{s\in\mathcal{S}}\sum_{\substack{k\in\mathcal{K}_s\\ c_{ij}^s\geq r_{q+2}}} N^s\pi^{sk}p_{ij}\delta_{ij}^{sk}$$
$$\leq \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}}\sum_{s\in\mathcal{S}}\sum_{\substack{k\in\mathcal{K}_s\\ c_{ij}^s\geq r_{q+1}}} N^s p_{ij}\delta_{ij}^{sk} \leq 1. \tag{49}$$

Consequently, we obtain

$$\Phi_\alpha(r_q;\pi) \leq \Phi_\alpha(r_{q+1};\pi) \leq \cdots \leq \Phi_\alpha(r_{M_{\mathcal{A}}};\pi). \tag{50}$$

Let π_q be an optimal solution for $f_\alpha(r_q) = \min_{\pi\in\Omega}\Phi_\alpha(r_q;\pi)$; that is $f_\alpha(r_q) = \Phi_\alpha(r_q;\pi_q)$. Then, we have

$$f_\alpha(r_q) = \Phi_\alpha(r_q;\pi_q) \leq \Phi_\alpha(r_q;\pi_{q+1}) \leq \Phi_\alpha(r_{q+1};\pi_{q+1}) = f_\alpha(r_{q+1}). \tag{51}$$

Similarly,

$$f_\alpha(r_q) \leq f_\alpha(r_{q+1}) \leq \cdots \leq f_\alpha(r_{M_{\mathcal{A}}}). \tag{52}$$

This completes the proof. □

Instead of considering all r values in Θ, we can narrow the searching range for r if there exist r values satisfying (45). Let $\hat{q}$ be the smallest index to satisfy (45). By Theorem 2, it is proved that $f_\alpha(r_{\hat{q}}) \leq f_\alpha(r_{\hat{q}+1}) \leq \cdots \leq f_\alpha(r_{M_{\mathcal{A}}})$; thus excluding $r \in \{\hat{q}+1,\cdots,r_{M_{\mathcal{A}}}\}$ to search the minimal $f_\alpha(r)$. The CVaR minimization model (42) can be rewritten as:

$$\min_{r\in\{r_0,r_1,\cdots,r_{\hat{q}}\}} f_\alpha(r) \tag{53}$$

If (45) is not satisfied for any q, every $r \in \Theta$ should be considered.

5 Computational Scheme for the CVaR Minimization Model

In this section, an efficient computational scheme to solve the CVaR minimization model for hazmat transportation network design is proposed. The proposed CVaR minimization model for network design is a nonlinear optimization model. Based on (42), the proposed network design model can be decomposed into two stages. At the first stage, we search r within a finite set. At the second stage, $f_\alpha(r)$ is solved to yield the network design solution.

$$f_\alpha(r) = \left\{\min_{\pi,y,z}\left[r + \frac{1}{1-\alpha}\sum_{(i,j)\in\mathcal{A}}\sum_{s\in\mathcal{S}}\sum_{k\in\mathcal{K}_s} N^s\pi^{sk}p_{ij}\delta_{ij}^{sk}\left[c_{ij}^s - r\right]^+\right] \text{ subject to (28)–(34).}\right\} \tag{54}$$

Because of the nonlinearity to link the route-choice probabilities and path availabilities in (32), we linearize as follows:

$$\sum_{l\in\mathcal{K}_s} \rho^{sl}\phi^{skl} = \rho^{sk} z^{sk}, \qquad \forall s \in \mathcal{S}, \forall k \in \mathcal{K}_s \tag{55}$$

$$\phi^{skl} \leq z^{sl}, \qquad \forall s \in \mathcal{S}, \forall k \in \mathcal{K}_s, \forall l \in \mathcal{K}_s \tag{56}$$

$$\phi^{skl} \geq -(1 - z^{sl}) + \pi^{sk}, \qquad \forall s \in \mathcal{S}, \forall k \in \mathcal{K}_s, \forall l \in \mathcal{K}_s \tag{57}$$

$$0 \leq \phi^{skl} \leq \pi^{sk}, \qquad \forall s \in \mathcal{S}, \forall k \in \mathcal{K}_s, \forall l \in \mathcal{K}_s. \tag{58}$$

The parameter ρ can be computed with (14). Then, $f_\alpha(r)$ is reformulated as a MILP problem.

Despite the fact that we may use Theorem 2 to reduce the searching set for r variable, it is still time-consuming to compute $f_\alpha(r)$ given all potential r values if the scale of a network is large. Finding the optimal r can be accelerated by developing an efficient search scheme which depends on $f_\alpha(r)$. Besides, solving $f_\alpha(r)$ is very difficult when many path alternatives are considered for a complicated network. Sometimes, it is even impractical to obtain a good feasible solution for $f_\alpha(r)$.

We propose a line search with mapping to obtain optimal r as shown in Section 5.1 and show that Benders decomposition can generate upper and lower bounds of MILPs for given r values thus solving the $f_\alpha(r)$ problem in Section 5.2. Generating useful lower bounds by Benders decomposition, however, costs large computation efforts while good upper bounds can be obtained after a certain number of iterations. In this case, we terminate the algorithm by some criteria and gain the best feasible solutions from upper bounds.

5.1 A Line Search with Mapping

To search the optimal r value for the proposed network design model, we only consider a narrowed range of values checked by Theorem 2. Initially, we can think of obtaining an optimal solution for network design problem by visiting every value in Θ. If Θ involves a large number of values, the computation for the problem can be time-consuming because we need to solve a large number of MILPs. A searching mechanism for r based on line search methods are proposed in order to solve the problem efficiently. We use the Golden Section method. When it is applied to a strictly quasiconvex function, the Golden Section method can find a global minimal solution. The essence of the Golden Section method is to reuse one searching point in previous iteration and compare with an updated point derived by the golden ratio to reduce computations. Note that the golden ratio is 0.618.

We use the same idea to develop a discrete version of the Golden Section method, which only evaluates a limited number of r values in Θ. Usually, a line section method minimizes a nonlinear optimization problem over the interval $[a_0, b_0]$. The optimal r value lies in Θ, so $a_0 = 0$ and b_0 would be the smallest r value satisfying (45) by Theorem 2.

A line search algorithm usually copes with a continuous variable from a certain interval. In the proposed model, optimal r value is from a finite set. We map the updated point in iterations to value in the finite set using a simple mechanism. The simple mechanism can guarantee the correctness of searching interval. The procedures for searching optimal r for the proposed model are shown in Algorithm 1.

Algorithm 1 A line search with mapping

1: ***Initialization:*** Check the largest q (q^*) which satisfies (45). Let $k \leftarrow 0$ and $a_k \leftarrow 0$, $b_k \leftarrow r_{q^*}$. $\lambda_k = a_k + (1-\varphi)(b_k - a_k)$ and $\mu_k = a_k + \varphi(b_k - a_k)$. Find the left-closest value to λ_k (λ_{left}) and the right-closest value to μ_k (μ_{right}) among Θ. Let $\lambda_k = \lambda_{\text{left}}, \mu_k = \mu_{\text{right}}$ and

$$f_\alpha(\lambda_k) = \min_{\pi \in \Omega} \Phi_\alpha(\lambda_k; \pi)$$
$$f_\alpha(\mu_k) = \min_{\pi \in \Omega} \Phi_\alpha(\mu_k; \pi)$$

2: ***Convergence check:*** If $a_k = r_q$ and $b_k = r_q$ or r_{q+1} for any $q = 0, 1, \cdots, (q^* - 1)$, go to Step 6; otherwise, continue estimating $f_\alpha(\lambda_k)$ and $f_\alpha(\mu_k)$. If $f_\alpha(\lambda_k) > f_\alpha(\mu_k)$, go to Step 3; if $f_\alpha(\lambda_k) \leq f_\alpha(\mu_k)$, go to Step 4.

3: ***Reuse μ_k:*** Find the right-closest value to λ_k in Θ (λ_{right}) and let $a_{k+1} = \lambda_{\text{right}}$ and $b_{k+1} = b_k$. If $\mu_k - a_{k+1} \leq b_{k+1} - \mu_k$, let

$$\lambda_{k+1} = \mu_k, f_\alpha(\lambda_{k+1}) = f_\alpha(\mu_k)$$
$$\mu_{k+1} = \tfrac{\mu_k + b_{k+1}}{2}.$$

Find the right-closest value to μ_{k+1} in Θ (μ_{right}) and let $\mu_{k+1} = \mu_{\text{right}}$. Evaluate $f_\alpha(\mu_{k+1})$.
If $\mu_k - a_{k+1} > b_{k+1} - \mu_k$, let

$$\mu_{k+1} = \mu_k, f_\alpha(\mu_{k+1}) = f_\alpha(\mu_k)$$
$$\lambda_{k+1} = \tfrac{a_{k+1} + \mu_k}{2}.$$

Find the left-closest value to λ_{k+1} in Θ (λ_{left}) and let $\lambda_{k+1} = \lambda_{\text{left}}$. Evaluate $f_\alpha(\lambda_{k+1})$.
Go to Step 5.

4: ***Reuse λ_k:*** Find the left-closest value to μ_k in Θ (μ_{left}) and let $a_{k+1} = a_k$ and $b_{k+1} = \mu_{\text{left}}$. If $\lambda_k - a_{k+1} \leq b_{k+1} - \lambda_k$, let

$$\lambda_{k+1} = \lambda_k, f_\alpha(\lambda_{k+1}) = f_\alpha(\lambda_k)$$
$$\mu_{k+1} = \tfrac{\lambda_k + b_{k+1}}{2}.$$

Find the right-closest value to μ_{k+1} in Θ (μ_{right}) and let $\mu_{k+1} = \mu_{\text{right}}$. Evaluate $f_\alpha(\mu_{k+1})$.
If $\lambda_k - a_{k+1} > b_{k+1} - \lambda_k$, let

$$\mu_{k+1} = \lambda_k, f_\alpha(\mu_{k+1}) = f_\alpha(\lambda_k)$$
$$\lambda_{k+1} = \tfrac{a_{k+1} + \lambda_k}{2}.$$

Find the left-closest value to λ_{k+1} in Θ (λ_{left}) and let $\lambda_{k+1} = \lambda_{\text{left}}$. Evaluate $f_\alpha(\lambda_{k+1})$.
Go to Step 5.

5: ***Iteration update:*** $k \leftarrow k + 1$ and go to Step 2.

6: ***Determine optimal solution:*** Evaluate for $f_\alpha(a_k)$ and $f_\alpha(b_k)$. If $f_\alpha(a_k) \leq f_\alpha(b_k)$, $r^* = a_k$; otherwise, $r^* = b_k$. Stop.

5.2 Benders Decomposition for $f_\alpha(r)$

The line search for r highly depends on obtaining optimal objective values for MILPs. As the size of the network increases, the computation time for solving $f_\alpha(r)$ given r goes up exponentially. We benefit from generating upper and lower bounds for $f_\alpha(r)$ and solving the problem iteratively. Seen from the structure of the MILPs, it is found that $f_\alpha(r)$ can be decomposed into: (1) optimizing network design (2) analyzing probabilities assigned for paths.

Benders decomposition is a popular algorithm framework to deal with complicating variables and large-scale optimization problems in which variables and constraints are decomposed into a master problem and subproblems. The algorithm employs cutting-planes procedures for the master problem based on subproblems until it converges. There are two categories of cuts in Benders composition. When a subproblem reaches an optimal solution but its optimal objective value is not consistent with the master problem's, an optimality cut based on dual of a subproblem is generated. On the other hand, a feasibility cut can be generated if a subproblem is infeasible. Taking advantage of the extreme ray for the dual of a infeasible subproblem can help to generate a feasibility cut. Theories and applications for Benders decomposition are developed widely. Geoffrion (1972) generalized Benders' approach to a broader class of programs in which parameterized subproblems need no longer be a linear program decades ago. Stochastic programming problems, which is well known as its stage structure can be solved efficiently by Benders decomposition (Santoso et al., 2005).

We implement Benders decomposition to solve MILPs and obtain $f_\alpha(r)$. The network design y and path availability z are master problem variables while the probabilities related variables including π and ϕ are in subproblems.

With Benders decomposition, we present the master problem as follows:

$$\text{(master)} \ \min_{g,y,z} \sum_{s\in S}\sum_{k\in\mathcal{K}_s} g^{sk} \tag{59}$$

$$\text{s.t. } (28)\text{–}(31), (33)\text{–}(34)$$

$$g^{s_t k_t} \geq \rho^{s_t k_t} z^{s_t k_t}\lambda_t + \sum_{l\in\mathcal{K}_{s_t}} z^{s_t l}\mu_t^l + \sum_{l\in\mathcal{K}_{s_t}} (-1+z^{s_t l})v_t^l, \qquad \forall t = 1,2,\cdots \tag{60}$$

where t denotes the number of cut generated by the t-th iteration of Benders decomposition. Constraints (60) are further explained by subproblem duals later.

The subproblems which analyze the route-choice probabilities (32) are decomposed by $s \in \mathcal{S}, k \in \mathcal{K}_s$ with dual variables $(\lambda, \mu^l, v^l, \omega^l)$ as follows:

$$\min_{\pi^{sk}} \sum_{(i,j)\in\mathcal{A}} N^s \pi^{sk} p_{ij}\delta_{ij}^{sk}\left[c_{ij}^s - r\right]^+ \tag{61}$$

$$\text{s.t. } \sum_{l\in\mathcal{K}_s} \rho^{sl}\phi^{skl} = \rho^{sk} z^{sk} \qquad (\lambda) \tag{62}$$

$$\phi^{skl} \leq z^{sl}, \qquad \forall l \in \mathcal{K}_s \qquad (\mu^l \leq 0) \tag{63}$$

$$\phi^{skl} \geq -(1-z^{sl}) + \pi^{sk}, \qquad \forall l \in \mathcal{K}_s \qquad (v^l \geq 0) \tag{64}$$

$$\phi^{skl} \leq \pi^{sk}, \qquad \forall l \in \mathcal{K}_s \qquad (\omega^l \leq 0) \tag{65}$$

$$\pi^{sk} \text{ free}, \tag{66}$$

$$\phi^{skl} \geq 0, \qquad \forall l \in \mathcal{K}_s \tag{67}$$

Feed with master problem variables, route-choice probabilities can be estimated from subproblems.

Therefore, subproblems are feasible making it only necessary to generate optimality cuts from subproblem duals. The subproblem dual is defined as follows:

$$(\mathsf{SD}^{sk})\ \hat{g}^{sk} = \max_{\lambda,\mu,v,\omega} \rho^{sk} z^{sk} \lambda + \sum_{l \in \mathcal{K}_s} z^{sl} \mu^l + \sum_{l \in \mathcal{K}_s} (-1 + z^{sl}) v^l \tag{68}$$

$$\text{s.t. } -\sum_{l \in \mathcal{K}_s} \mu^l - \sum_{l \in \mathcal{K}_s} v^l = \sum_{(i,j) \in \mathcal{A}} N^s p_{ij} \delta_{ij}^{sk} \left[c_{ij}^s - r\right]^+ \tag{69}$$

$$\rho^{sl} \lambda + \mu^l + v^l + \omega^l \leq 0, \forall l \in \mathcal{K}_s. \tag{70}$$

In subproblem duals, we can obtain a (s_t, k_t) with the objective value $\hat{g}^{s_t k_t}$ and the solution $(\lambda_t, \mu_t^l, v_t^l, \omega_t^l)$ accordingly. Let $\tilde{g}^{s_t k_t}$ be an optimal solution for the master problem. If $\hat{g}^{s_t k_t}$ is greater than $\tilde{g}^{s_t k_t}$, an optimality cut can be generated as (60) using (68). The algorithm is summarized in Algorithm 2. In Algorithm 2, ϵ is a small positive parameter. Besides, I is used to indicate

Algorithm 2 Benders decomposition for $f_\alpha(r)$

1: ***Initialization:*** Set $t = 0$, upper bound $\mathsf{UB} = \infty$ and lower bound $\mathsf{LB} = 0$. Go to Step 2.
2: ***Solve master problem:*** Solve the master problem and obtain the optimal solution $(\tilde{g}, \tilde{y}, \tilde{z})$. Let $\mathsf{LB} = \sum_{s \in S} \sum_{k \in \mathcal{K}_s} \tilde{g}^{sk}$ and $I = 0$. Go to Step 3.
3: ***Solve subproblem:*** For (s, k), solve SD^{sk} problem based on $\tilde{z}$ and obtain optimal solution $(\hat{\lambda}, \hat{\mu^l}, \hat{v^l}, \hat{\omega^l})$. The optimal objective value for the subproblem is $\hat{g}^{sk}$. If all (s, k) are visited, go to Step 5; otherwise, go to Step 4.
4: ***Generate an optimality cut:*** If $I = 1$ go to Step 2; otherwise, compare $\tilde{g}^{sk}$ and $\hat{g}^{sk}$. If $\hat{g}^{sk} - \tilde{g}^{sk} \geq \epsilon$, update $I \leftarrow 1, t \leftarrow t + 1, s_t \leftarrow s, k_t \leftarrow k, \lambda_t \leftarrow \tilde{\lambda}, \mu_t \leftarrow \tilde{\mu}, v_t \leftarrow \tilde{v}$ and an optimality cut is generated; otherwise, update (s, k) and go to Step 3.
5: ***Convergence check:*** If $\mathsf{UB} > \sum_{s \in S} \sum_{k \in \mathcal{K}_s} \hat{g}^{sk}$, set $\mathsf{UB} = \sum_{s \in S} \sum_{k \in \mathcal{K}_s} \hat{g}^{sk}$. If $\mathsf{UB} - \mathsf{LB} \leq \epsilon$, terminate; otherwise go to Step 1.

whether an optimality cut is generated. Based on an optimal solution for the master problem, we can generate multiple optimality cuts from different subproblems. The master problem becomes very difficult to solve if too many cuts are added at a time. If a master problem costs huge computations at the beginning, it would be hard to yield an upper bound. In order to produce upper bounds effectively, we only add one optimality cut after solving the master problem until the algorithm terminates.

We implement Benders decomposition on the Ravenna network in (Bonvicini and Spadoni, 2008; Erkut and Gzara, 2008) with 105 nodes and 134 undirected arcs. Four kinds of hazardous materials are considered including methanol, chlorine, gasoline and LPG. There are 31 shipments and each shipment defines a certain demand of a hazmat transported from an OD pair. For each shipment, we generate 50 paths using k-shortest path approach to test the performance of the proposed framework. The computation process for solving $f_{0.95}(0.454)$ is shown in Figure 1. We terminate the algorithm when the optimality gap is less than 5%. In this example, we can see that a good feasible solution is achieved within a small number of iterations. The improvement of lower bound, however, is very slow. Besides, it becomes more difficult to solve the master problem as iteration proceeds. It

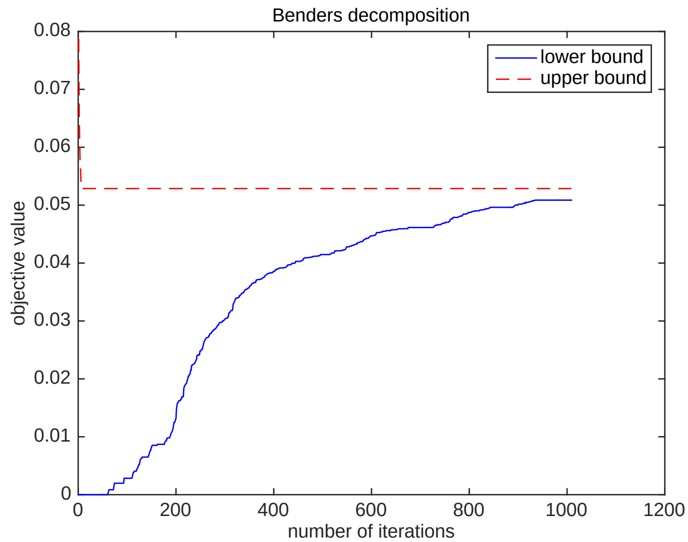


Figure 1: Lower bounds and upper bounds for a MILP given $r = 0.454$ and $\alpha = 0.95$ by Benders decomposition for the Ravenna network.

indicates that the time spent on the iteration close to the optimal solution can be far more than early iterations. An optimal solution is obtained when the upper bound and the lower bound are close.

Since we can obtain feasible solutions and useful upper bounds before reaching the convergence of Benders decomposition, a close optimal solution generated by a set of feasible solutions is used. When the upper bound does not improve in the next a few iterations, we terminate the algorithm. The local optimality can be guaranteed for the best feasible solution thus providing a practical approach. Besides, the effectiveness to terminate at a good feasible solution for $f_\alpha(r)$ accelerates the solving process.

5.3 Performance of Algorithm 1 Depending on Algorithm 2

This section discusses the performance of Algorithm 1 depending on Algorithm 2. The Ravenna network with 20 paths for each shipment are used for experiments in this section. Let $\alpha = 0.95$ and the maximum number of closed arcs $N = 10$. To solve the proposed CVaR minimization model for hazmat network design, we incorporate the searching scheme for r in Section 5.1 with evaluations of $f_\alpha(r)$ using Algorithm 2. We can either use the optimal or the best feasiable solution obatained from Algorithm 2 to proceed the searching of r in Algorithm 1. The searching process of r is shown in Figure 2. It can be seen that an optimal network design is achieved when solving MILP with $r = 0.687$ and the minimum risk equals to 0.732. The line search with mapping efficiently identifies an optimal r value if we can obtain $f_\alpha(r)$. In Figure 2, it is found that the optimal r value is 0.699 and the approximated minimum risk is 0.742 by the best feasible solution of $f_\alpha(r)$. Accordingly, the network design results are shown in Figure 3. The number of closed arcs in both cases is 10

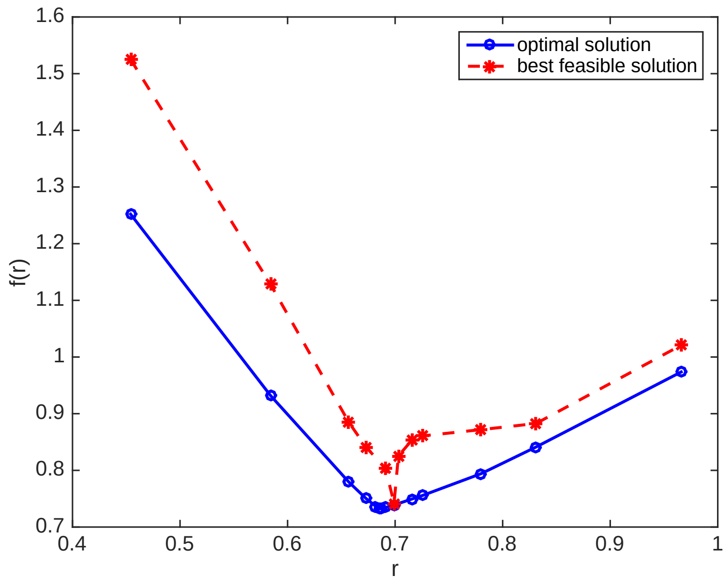


Figure 2: Searching of r based on the optimal and the best feasible solution by Algorithm 2

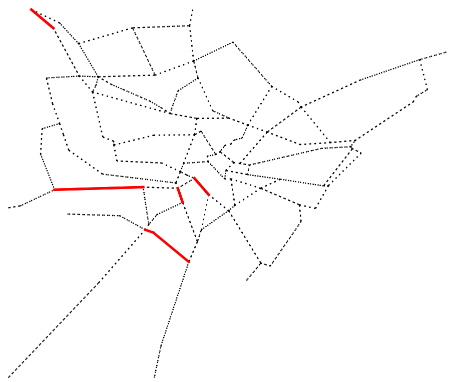
with 8 of which are the same. In the optimal design, arc $(56, 69)$ and $(66, 83)$ should be closed while our proposed approach determines arc $(5, 7)$ and $(7, 5)$ are closed. Given the best feasible design, the CVaR is the minimum value for $\Phi(r; \pi)$ through all r values. Hence, the risk for best feasible network design is less than or equal to 0.742.

Our proposed method yields a network design with the risk no higher than 1.35% of the global optimal solution making it acceptable for decision makers. In addition, it cost 3 hours to compute the optimal hazmat network design depending on exact value of $f_\alpha(r)$ while the best feasible design is obtained in 1 hour and 33 minutes. Therefore, our proposed computational scheme to incorporate the line search for r with a best feasible solution for $f_\alpha(r)$ is very efficient and effective.

6 Numerical Experiments

In this section, applications of the proposed model are shown. All numerical experiments in this section are conducted using the Ravenna (Bonvicini and Spadoni, 2008; Erkut and Gzara, 2008), the Albany (Kang et al., 2014) and the Buffalo (Toumazis and Kwon, 2016) networks. Four kinds of hazardous materials are considered: methanol, chlorine, gasoline and LPG. There are 31 shipments transported through the networks. The data set includes the length of each arc, the population that each kind of hazmat can influence on each arc, the OD pairs for each kind of hazmat and the demand of hazmat accordingly.

Our proposed model is a path-based hazmat network design model which requires specified path alternatives by hazmat carriers. One of the most typical approaches to generate path set is k shortest path algorithm. Yen (1971) first presented an algorithm to find the loopless k shortest



(a) An optimal solution

(b) The best feasible solution

Figure 3: Network designs based on the optimal and the best feasible solution by Algorithm 2

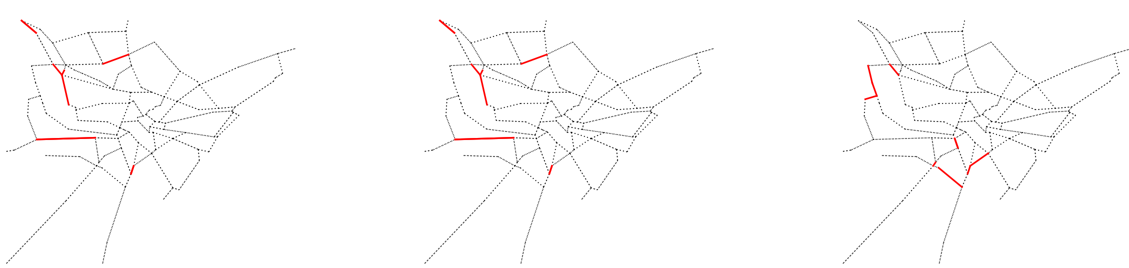
length path. Despite the modifications or improvements of k shortest path algorithm, this approach rarely emphasizes on accident consequences of arcs. If the set of path alternatives obtained by k shortest path algorithm is very small, for example, only five paths for each shipment, some important arcs with high chosen probabilities and high risks can be left out. On the other hand, it is nearly impossible to solve our proposed model enumerating all paths for all shipments due to the tremendous model size. Hazmat carriers can be restricted to some roads due to massive weights, large heights and long lengths for trucks. Usually, hazmat carriers select a route within a limited number of path alternatives. We use k shortest path algorithm to enumerate a list of paths which consider the shortest 50 paths for each shipment preparing for the proposed hazmat network design model.

The computational scheme in Section 5 is coded in the Julia Language with the JuMP.jl package (Dunning et al., 2017) and CPLEX solver of version 12.6 is used. The experiments are implemented on a computer with 8GM of RAM and a 2.7GHz processor.

Table 2 shows the computation time and objective values using Algorithm 1 and Algorithm 2. To accelerate the computation, Algorithm 2 is terminated when the solution is not improved within the next fifty iterations. For the Ravenna network, the road-ban decisions with different confidence levels can be seen in Figure 4. When $\alpha = 0.900$ and $\alpha = 0.950$, the optimal network designs are the same. Regulators for hazmat transportation have different attitudes towards risks but may end up with the same optimal network design. Theoretically, the proposed model indicates that high confidence level α considers more on severer accidence consequences than low confidence level does. With the increasing of confidence level, the optimal network design for the proposed model can vary a lot. The optimal network design of $\alpha = 0.990$ only has one common closed arc – arc $(78, 74)$ with $\alpha = 0.900$ and $\alpha = 0.950$. For example, closing arc $(3, 6)$ plays a significant role in reducing risk

Table 2: Numerical results for different networks

Network	α	Computation time	CVaR_α
Buffalo	0.900	2hr 21min	708.6
	0.950	2hr 4min	875.6
	0.990	2hr 23min	1127.5
	0.999	3hr 8min	1304.6
Ravenna	0.900	2hr 53min	762.4
	0.950	2hr 57min	837.8
	0.990	2hr 44min	1157.3
	0.999	2hr 4min	1487.0
Albany	0.900	1hr 49min	504.1
	0.950	2hr 14min	679.7
	0.990	1hr 52min	844.6
	0.999	2hr 13min	1115.7



(a) $\alpha = 0.900$ (b) $\alpha = 0.950$ (c) $\alpha = 0.990$

Figure 4: Ravenna road-ban with different α values

Table 3: Comparions of algorithms to obtain $f_\alpha(r) = r + \frac{1}{1-\alpha}h(r)$ for the Ravenna network

r value ($\times 10^3$ population exposure)	$h(r)$		$\frac{h(r)_{\text{CPLEX}} - h(r)_{\text{Algorithm 2}}}{h(r)_{\text{Algorithm 2}}}$
	Algorithm 2	CPLEX	
0.454	0.046	0.079	72.8 %
0.265	0.091	0.152	67.7 %
0.427	0.053	0.086	64.2 %
0.130	0.167	0.255	52.9 %
0.352	0.068	0.110	62.7 %
0.199	0.120	0.191	59.3 %
0.165	0.133	0.213	60.7 %
0.231	0.109	0.171	57.8 %
0.183	0.124	0.201	61.4 %
0.174	0.130	0.207	59.3 %
0.191	0.121	0.196	62.1 %
0.197	0.122	0.192	57.7 %
0.186	0.123	0.199	61.1 %
0.193	0.123	0.194	58.1 %
0.189	0.123	0.197	59.8 %
0.188	0.123	0.198	60.7 %

with $\alpha = 0.990$ but not in $\alpha = 0.900$ and $\alpha = 0.950$ cases. If we close arc $(3, 6)$, the large accident consequences by hazmat within 1% chance to happen can be avoided while it may not be effective to reduce the risk brought by 10% potential hazmat truck accidents.

6.1 Comparisons for Algorithms

To solve the CVaR minimization network design model, Algorithms 1 and 2 are proposed. Algorithm 1 and the CPLEX solver can also be used to solve the problem. Algorithm 2 and the CPLEX solver are compared for the values of $f_\alpha(r)$ obtained. CPLEX may not return an optimal solution for $f_\alpha(r)$ within limited time. We have restricted 30 minutes as time limit for both Algorithm 2 and CPLEX. Seen from (54), $f_\alpha(r)$ can be estimated by solving an MILP which only relates to r. Let $f_\alpha(r) = r + \frac{1}{1-\alpha}h(r)$ and $h(r)$ denote the objective value of the MILP. Given r values, Table 3 shows the obtained $h(r)$ by using Algorithm 2 and CPLEX solver. It can be seen that Algorithm 2 performs better than CPLEX in solving the MILPs and obtaining $f_\alpha(r)$. Algorithm 2 can provide acceptable solutions for $f_\alpha(r)$ thus proceeding Algorithm 1.

6.2 Comparisons of Models

To show the value of our model, comparisons of the deterministic model described in Section 2 and the proposed model for the Ravenna network are conducted. The results are shown in Table 4. When $\alpha = 0.900$ and $\alpha = 0.950$, the proposed model generates optimal network designs which are similar to the optimal solution by the deterministic model. If the decision makers and regulators for

Table 4: Comparisons of CVaR and ER model for the Ravenna network

Model	Confidence level α	CVaR_α 0.900	0.950	0.990	0.999	ER
CVaR min.	0.900	762.4	–	–	–	369.2
	0.950	–	837.8	–	–	369.2
	0.990	–	–	1157.3	–	401.5
	0.999	–	–	–	1487.0	397.9
Determin.		768.7	850.5	1257.6	1687.0	358.1

the hazmat transportation network have relatively low confidence levels and pay a limited attention on severe accidents, the network design by the proposed model is close to the deterministic model. When $\alpha = 0.990$, the CVaR of network design by the deterministic model would be 8.7% higher compared to the proposed model while the ER is 12.1% lower. If $\alpha = 0.999$, the CVaR of network design from the deterministic model would be 13.45% larger than the proposed model while the ER is 10% lower. It can be found that the difference of hazmat transportation network design between the proposed model and the deterministic model becomes more significant as the confidence level increases. The proposed model emphasizes a hazmat transport network design which cannot be addressed by the deterministic model in some cases.

In the Ravenna network, the optimal network designs by the proposed model and the deterministic model yield different available paths for shipments with which lead to different risks. The comparisons of available paths for transporting methanol from node 110 to node 105 by both models are shown in Table 5. For each path, the length, the ER to transport methanol and the probability to be chosen by hazmat carriers are given. It is found that two available paths are the same while the rest of them are different either in length or ER for both models. The deterministic model addresses more on short length paths than the proposed model while ignoring the risk and the chosen probability for each path. In Table 5, the shortest path with the highest probability by both models is path 1. Since the deterministic model generates shorter paths than the proposed model thus making some paths comparable and decreasing the probability of choosing path 1. Although the minimum ER path is path 1 for both models, the higher chosen probability of path 1 in the proposed model results in lower risk for the shipment. It is reasonable that the proposed model is preferred to capture risk with considerations of probabilistic routes and protect the road network from severe consequences.

7 Concluding Remarks

In this paper, we consider a probabilistic route choice model to analyze hazmat carriers' behavior in response to a network design in hazardous materials transportation. With the probabilistic route choice, the risk distribution for hazmat transportation incorporates with not only the road accident probability but also the carriers routing behavior. Followed by Toumazis et al. (2013), we introduce conditional value-at-risk (CVaR) as a general, coherent and risk-averse approach. We present a

Table 5: Comparisons of available paths for transporting methanol from node 110 to node 105 between CVaR and the deterministic model for the Ravenna network

Model		Path	Length	ER	Prob
CVaR min.	1:	110 → 104 → 83 → 78 → 62 → 54 → 45 → 31 → 98 → 14 → 11 → 6 → 3 → 5 → 105	18.24	0.0149	0.345
	14 :	110 → 104 → 83 → 78 → 62 → 54 → 45 → 43 → 36 → 30 → 26 → 24 → 20 → 10 → 5 → 105	26.04	0.0169	0.158
	19 :	110 → 104 → 83 → 78 → 62 → 54 → 56 → 46 → 43 → 36 → 30 → 26 → 24 → 20 → 10 → 5 → 105	26.87	0.0168	0.146
	26 :	110 → 104 → 83 → 78 → 62 → 54 → 45 → 43 → 36 → 30 → 26 → 24 → 20 → 10 → 9 → 7 → 5 → 105	28.17	0.0172	0.128
	34 :	110 → 104 → 83 → 78 → 62 → 54 → 56 → 46 → 43 → 36 → 30 → 26 → 24 → 20 → 10 → 9 → 7 → 5 → 105	29.00	0.0172	0.118
	44 :	110 → 104 → 83 → 78 → 62 → 54 → 56 → 46 → 43 → 45 → 31 → 98 → 14 → 11 → 6 → 3 → 5 → 105	30.18	0.0171	0.105
Determin.	1 :	110 → 104 → 83 → 78 → 62 → 54 → 45 → 31 → 98 → 14 → 11 → 6 → 3 → 5 → 105	18.24	0.0149	0.246
	3 :	110 → 104 → 83 → 78 → 62 → 57 → 58 → 38 → 54 → 45 → 31 → 98 → 14 → 11 → 6 → 3 → 5 → 105	19.93	0.0169	0.207
	5 :	110 → 104 → 83 → 78 → 74 → 76 → 69 → 56 → 54 → 45 → 31 → 98 → 14 → 11 → 6 → 3 → 5 → 105	23.26	0.0157	0.149
	8 :	110 → 104 → 83 → 78 → 74 → 69 → 56 → 54 → 45 → 31 → 98 → 14 → 11 → 6 → 3 → 5 → 105	24.77	0.0160	0.128
	18 :	110 → 104 → 83 → 78 → 74 → 76 → 69 → 56 → 46 → 43 → 45 → 31 → 98 → 14 → 11 → 6 → 3 → 5 → 105	26.69	0.0164	0.106
	27 :	110 → 104 → 83 → 78 → 74 → 69 → 56 → 46 → 43 → 45 → 31 → 98 → 14 → 11 → 6 → 3 → 5 → 105	28.20	0.0167	0.091
	44 :	110 → 104 → 83 → 78 → 62 → 54 → 56 → 46 → 43 → 45 → 31 → 98 → 14 → 11 → 6 → 3 → 5 → 105	30.18	0.0171	0.073

CVaR minimization model for network design. The proposed model is a nonlinear programming which can be decomposed into two stages: (1) searching the optimal solution for a nonnegative variable; (2) solving MILPs given the nonnegative variable. In applications, estimating a large number of MILPs is inefficient. Besides, solving a single MILP costs numerous computation efforts when a network is complicated. Therefore, we develop a line search with mapping based on Benders decomposition and obtain quality network design solutions.

We present case studies in the real road networks of Ravenna, Albany, and Buffalo. The comparisons of algorithms show that the proposed methods can solve the CVaR minimization network design problem efficiently and generate quality network design solutions. To highlight the value of our model, comparisons of the deterministic model and the proposed model are conducted. When the confidence level of CVaR is small, it indicates that decision makers and regulators for the transportation network pay limited attention on sever accidents. The hazmat network designs by the deterministic model and our proposed model are similar. Our proposed model, however, can consider the network designs with high risk-averse attitudes. In addition, we model the uncertainty of route choices instead of using the shortest path to predict the behavior of hazmat carriers. Hence, our model can protect the road network from undesirable route-choices that may lead to severe consequences.

Acknowledgment

This manuscript is based upon work funded partially by a grant from the U.S. Department of Transportation's University Transportation Centers Program. However, the U.S. Government assumes no liability for the contents or use thereof. The contents of this report reflect the views of the authors, who are responsible for the facts and the accuracy of the information presented herein.

References

Abkowitz, M. D., M. Lepofsky, P. Cheng. 1992. Selecting criteria for designating hazardous materials highway routes. *Transportation Research Record* (1333).

Artzner, P., F. Delbaen, J.-M. Eber, D. Heath. 1997. Thinking coherently: Generalised scenarios rather than VaR should be used when calculating regulatory capital. *Risk-London-Risk Magazine Limited* **10** 68–71.

Artzner, P., F. Delbaen, J.-M. Eber, D. Heath. 1999. Coherent measures of risk. *Mathematical Finance* **9**(3) 203–228.

Ben-Akiva, M., M. Bierlaire. 1999. Discrete choice methods and their applications to short term travel decisions. *Handbook of Transportation Science*. Springer, 5–33.

Bonvicini, S., G. Spadoni. 2008. A hazmat multi-commodity routing model satisfying risk criteria: A case study. *Journal of Loss Prevention in the Process Industries* **21**(4) 345–358.

Cascetta, E., A. Nuzzolo, F. Russo, A. Vitetta. 1996. A modified logit route choice model overcoming path overlapping problems: specification and some calibration results for interurban networks. *Proceedings of the 13th International Symposium on Transportation and Traffic Theory*. Pergamon Oxford, NY, USA, 697–711.

Daganzo, C. F., Y. Sheffi. 1977. On stochastic models of traffic assignment. *Transportation Science* **11**(3) 253–274.

Duffie, D., J. Pan. 1997. An overview of value at risk. *The Journal of Derivatives* **4**(3) 7–49.

Dunning, I., J. Huchette, M. Lubin. 2017. Jump: A modeling language for mathematical optimization. *SIAM Review* **59**(2) 295–320. doi: 10.1137/15M1020575.

Erkut, E., F. Gzara. 2008. Solving the hazmat transport network design problem. *Computers & Operations Research* **35**(7) 2234–2247.

Federal Motor Carrier Safety Administration. 2016a. Cargo tank truck rollover prevention. URL `https://www.fmcsa.dot.gov/rolloverprevention`.

Federal Motor Carrier Safety Administration. 2016b. Federal motor carrier safety regulations. URL `https://www.fmcsa.dot.gov/regulations/hazardous-materials/how-comply-federal-hazardous-materials-regulations`.

Fontaine, P., S. Minner. 2018. Benders decomposition for the hazmat transport network design problem. *European Journal of Operational Research* .

Geoffrion, A. M. 1972. Generalized benders decomposition. *Journal of Optimization Theory and Applications* **10**(4) 237–260.

Gzara, F. 2013. A cutting plane approach for bilevel hazardous material transport network design. *Operations Research Letters* **41**(1) 40–46.

Jin, H., R. Batta. 1997. Objectives derived form viewing hazmat shipments as a sequence of independent Bernoulli trials. *Transportation Science* **31**(3) 252–261.

Kang, Y., R. Batta, C. Kwon. 2014. Value-at-risk model for hazardous material transportation. *Annals of Operations Research* **222**(1) 361–387.

Kara, B. Y., V. Verter. 2004. Designing a road network for hazardous materials transportation. *Transportation Science* **38**(2) 188–196.

McFadden, D. 1975. The revealed preferences of a government bureaucracy: Theory. *The Bell Journal of Economics* 401–416.

Pipeline and Hazardous Materials Safety Administration. 2016. Hazmat incident report: 10 year incident summary reports. URL `https://hip.phmsa.dot.gov/analyticsSOAP/`

saw.dll?Dashboard&NQUser=HazmatWebsiteUser1&NQPassword=HazmatWebsiteUser1&PortalPath=/shared/Public%20Website%20Pages/_portal/10%20Year%20Incident%20Summary%20Reports.

Prashker, J. N., S. Bekhor. 2004. Route choice models used in the stochastic user equilibrium problem: a review. *Transport Reviews* **24**(4) 437–463.

Ramming, M. S. 2001. Network knowledge and route choice. Ph.D. thesis, Massachusetts Institute of Technology.

Rockafellar, R. T., S. Uryasev. 2000. Optimization of conditional value-at-risk. *Journal of Risk* **2** 21–42.

Rockafellar, R. T., S. Uryasev. 2002. Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance* **26**(7) 1443–1471.

Santoso, T., S. Ahmed, M. Goetschalckx, A. Shapiro. 2005. A stochastic programming approach for supply chain network design under uncertainty. *European Journal of Operational Research* **167**(1) 96–115.

Sheffi, Y. 1985. *Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods*. Prentice-Hall, Englewood Cliffs, NJ.

Sun, L., M. H. Karwan, C. Kwon. 2016a. Incorporating driver behaviors in network design problems: Challenges and opportunities. *Transport Reviews* **36**(4) 454–478.

Sun, L., M. H. Karwan, C. Kwon. 2016b. Robust hazmat network design problems considering risk uncertainty. *Transportation Science* **50**(4) 1188–1203.

Sun, L., M. H. Karwan, C. Kwon. 2018. Generalized bounded rationality and robust multi-commodity network design. *Operations Research* **66**(1) 42–57.

Toumazis, I., C. Kwon. 2013. Routing hazardous materials on time-dependent networks using conditional value-at-risk. *Transportation Research Part C: Emerging Technologies* **37** 73–92.

Toumazis, I., C. Kwon. 2016. Worst-case conditional value-at-risk minimization for hazardous materials transportation. *Transportation Science* **50**(4) 1174–1187.

Toumazis, I., C. Kwon, R. Batta. 2013. Value-at-risk and conditional value-at-risk minimization for hazardous materials routing. *Handbook of OR/MS Models in Hazardous Materials Transportation*. Springer, 127–154.

Verter, V., B. Y. Kara. 2008. A path-based approach for hazmat transport network design. *Management Science* **54**(1) 29–40.

Yen, J. Y. 1971. Finding the k shortest loopless paths in a network. *Management Science* **17**(11) 712–716.