Generalized Bounded Rationality and Robust Multi-Commodity Network Design

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Often network users are not perfectly rational, especially when they are satisficing—rather than optimizing—decision makers and each individual's perception of the decision environment reflects personal preferences or perception errors due to lack of information. While the assumption of satisficing drivers has been used in modeling route choice behavior, this research uses a link-based perception error model to describe driver's uncertain behavior, without assuming stochasticity. In congestion-free networks, we show that the perception error model is more general than the existing bounded rationality models with satisficing drivers with special cases when the two approaches yield the same results; that is, satisficing under accurate perception is equivalent to optimizing under inaccurate perception. This motivates us to define generalized bounded rationality in route choice behavior modeling. The proposed modeling framework is general enough to capture link-specific cost-perception of drivers. We use a Monte Carlo method to estimate modeling parameter values to guarantee a certain coverage probability in comparison with the random utility model. We demonstrate how the notion of generalized bounded rationality can be used in robust multi-commodity network design problems and devise a cutting plane algorithm. We illustrate our approaches in the context of hazardous materials transportation.

Key words: bounded rationality; satisficing; perception; network design; robust optimization; inverse optimization

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1. Introduction

An economic, or (perfectly) *rational*, person needs to have a clear system of preferences and knowledge of the decision environment (Simon 1955). This *rationality assumption* has been widely used in many decision-making process modeling contexts. Due to various reasons, such as the lack of full information about environments surrounding decision-making and strong personal preferences that are difficult to measure and rarely taken into account, people often make decisions that cannot be justified by the rationality assumption. Instead, decision makers have limited information about surrounding environments and themselves, and hence are *boundedly rational*, instead of being *perfectly rational*. As opposed to the definition of a rational person by Simon (1955), the main two sources of boundedness of rationality may be: (1) *satisficing* rather than optimizing, and (2) errors in each individual's *perception* of the decision environment. Satisficing decision makers choose an alternative whose reward is above a certain threshold, called an aspiration level, or close enough to the best possible reward (Simon 1955). In the context of path finding in transportation networks, this paper shows that satisficing under accurate perception is equivalent to optimizing under inaccurate perception; hence the two sources may be modeled in a unified framework.

Since Simon (1955, 1956, 1959) tried to simplify the rationality assumption and understand human behavior in decision making, the notion of bounded rationality, especially satisficing behavior, has been used in modeling decision-making processes. Charnes and Cooper (1963) model satisficing behavior by maximizing the probability of achieving a certain reference profit value in the framework of chance-constraint stochastic optimization, and further show that such modeling is closely related to minimizing the deviation from the reference profit value. Cassidy et al. (1972) consider a similar modeling approach in random payoff games. In the context of risk management, Brown and Sim (2009) introduce the notion of satisficing measures and connect with other risk measures.

In the context of route choice, there is evidence that drivers are not perfectly rational (Nakayama et al. 2001, Zhu and Levinson 2010); hence they do not always choose the shortest path. To model such behavior of drivers, Mahmassani and Chang (1987) first used the assumption of boundedly rational drivers: drivers choose any path whose travel time is within a certain threshold, even if the path is sub-optimal. This assumption is related to the first source—satisficing behavior—of boundedness in rationality. Many other researchers followed such an assumption of satisficing drivers: in boundedly rational user equilibrium (BRUE) (Guo and Liu 2011, Di et al. 2013, 2014), in BRUE in dynamic settings (Wu et al. 2013, Szeto and Lo 2006, Han et al. 2015), and in congestion pricing (Lou et al. 2010, Di et al. 2016). See also review papers by Zhang (2011) and Sun et al. (2016b).

Another, more traditional stream of modeling route-choice behavior is based on random utility theory and discrete choice models (Sheffi 1985, Ben-Akiva and Lerman 1985). Travelers are assumed to maximize their utility and we assume that we as the modeler have some limited information on their utility function. As a result, drivers' utility involves a random error term, which is called a random *unobservable* utility component as opposed to the observable utility component that is known to the modeler. Assuming the random error terms follow certain distributions (for example, Gumbel distribution), we can determine the choice probability of each path over all other paths, with the assumption that a path is chosen if its utility is the maximum among all paths. While the random utility model (RUM) is somewhat related to the second source, note that the random component is not a perception *error* of the driver's utility, but his personal utility that is *unknown* to the modeler. It may, however, be interpreted as the driver's perception error from the modeler's perspective. Whether it is perception error or unobservable utility, RUM assumes that drivers are optimizing.

Although RUM and our perception error model are related, there exist fundamental differences. Instead of the unobservable utility components as in RUM, we use the notion of *perception error*. We assume that drivers *perceive* link travel time differently from the actual travel time, whatever the reason is. Based on their own perceived link travel times, drivers are assumed to make an optimal decision in our perception-error models. A major difference between the perception error and the unobservable random utility component is that the perception error belongs to a bounded set instead of being a random variable with a probability distribution. As we will see in this paper, the perception error model produces satisficing paths. We know that the resulting paths are candidate paths that *may* be chosen; but we neither know nor assume with what probability each path would be chosen, while we know the probability distribution of path choices in RUM.

We note that the relation between RUM and our perception error model resembles the relation between stochastic programming and robust optimization. In the theory of robust optimization (Ben-Tal and Nemirovski 2002, Bertsimas et al. 2011, Gabrel et al. 2014), one typically addresses uncertainty—as opposed to stochasticity—without assuming probabilistic distributions (Bertsimas and Sim 2004, 2003, Ben-Tal and Nemirovski 1998), or with partial distributional information (Jaillet et al. 2016, Delage and Ye 2010, Goh and Sim 2010, Wiesemann et al. 2014). In our perception error model, we follow the approach without probabilistic distribution information.

In multi-commodity networks without congestion effects, this paper aims to connect these two seemingly different modeling paradigms and shows that optimizing drivers with perception error and satisficing drivers without perception error make the same route decisions. With this finding, we propose the notion of *generalized bounded rationality* using the perception-error models. The generalization comes in two ways. First, our perception-error model provides a unified modeling framework that covers both sources of boundedness in rationality. Second, we show that our perception-error models are equivalent to some special cases of and more general than the existing bounded rationality models that assume satisficing behavior.

We emphasize that in the current literature, especially in transportation science and engineering, the consideration of bounded rationality is limited to the first source of bounded rationality—satisficing behavior. This paper first proposes a model that considers the second source—perception error—only and shows that the perception error model is more general than the satisficing models. In this paper, we clearly distinguish satisficing behavior from bounded rationality: satisficing behavior as a sub-concept of bounded rationality. This paper provides a framework that can model both satisficing behavior and perception error in bounded rationality, as opposed to the notion of a 'rational man' defined by Simon (1955).

We also elaborate various types of bounded rationality assumptions in the process. We first define *satisficing paths*, instead of the shortest path, and then distinguish *multiplicative* satisficing from *additive* satisficing. We also consider a new notion of *subpath satisficing*, in analogy to the subpath optimality property of the shortest path. We also demonstrate how one can define a generalized bounded rationality model using an *ellipsoidal* set of uncertain perception errors. To determine parameter values in the perception-error model that determine the size of the rationality bound, we investigate how the lengths of available paths between an origin-destination pair are distributed. To avoid full path enumeration, which is a numerically challenging task, we use a Monte Carlo method to estimate the path length distribution.

In this paper, we relate drivers' route choice behaviors to a particular uncertainty parametric set, using a set of uncertain perception errors. The notion of an uncertainty set originates from the literature of robust optimization (Bertsimas et al. 2011). It seeks to find a solution that satisfies the realization of all the scenarios represented by the uncertainty set, that is, it optimizes the worst case scenario. The uncertainty set can be represented by an ellipsoidal set (Ben-Tal and Nemirovski 2002, El Ghaoui et al. 1998), a cardinality constrained uncertainty set (Bertsimas and Sim 2003, 2004, Kwon et al. 2013), or a norm uncertainty set (Bertsimas et al. 2004). In this research, we show that a similar modeling approach is possible in studying uncertain route-decision making of drivers with bounded rationality.

We emphasize that our generalized bounded rationality model based on perception-error is link-based, while many existing satisficing or bounded rationality models are path-based with one exception (Lou et al. 2010). Although we are uncertain about each driver's overall preferences, we may be certain for some road links. For example, all drivers perceive the travel time very precisely in some road links, but drivers perceive the travel time with a greater difference in some other road links. Existing satisficing models have little flexibility to fully capture these link specific preferences, or perception errors.

Models of route-decisions are most useful in network design problems, where a central authority predicts drivers' behavior according to network policy changes. We illustrate how perception-error models can be used in robust multi-commodity network design problems and propose a cutting plane algorithm. Numerical examples are provided in the context of hazardous materials transportation.

While the findings in this paper are limited to path finding behavior in transportation networks, it seems that our main result—satisficing under accurate perception is equivalent to optimizing under inaccurate perception—has potential to generalize the bounded rationality approaches in other areas of decision-making modeling.

2. Bounded Rationality: Satisficing Behavior and Perception Error

We consider a directed graph $G(\mathcal{N},\mathcal{A})$ where $\mathcal{N}$ is the set of nodes and $\mathcal{A}$ is the set of links. We consider a single origin-destination pair o and d, and we define the set of path vectors from o to d:

$$\mathbb{X} = \left\{ x : \sum_{(i,j)\in\mathcal{A}} x_{ij} - \sum_{(j,i)\in\mathcal{A}} x_{ji} = b_i \ \forall i \in \mathcal{N}, \quad x_{ij} \in \{0,1\} \ \forall (i,j) \in \mathcal{A} \right\} \tag{1}$$

where $b_i = 1$ for $i = o$, $b_i = -1$ for $i = d$, and $b_i = 0$ for all other nodes $i \in \mathcal{N}$. For each link $(i,j) \in \mathcal{A}$, the link travel cost is c_{ij}.

In the context of shortest-path finding, a driver with *(perfect) rationality* chooses the shortest path based on the exact value of travel cost vector c. That is, rationality corresponds to both optimizing behavior and full information on the network. Let us denote the length of the shortest-path by c^0 and a vector of the shortest-path by x^0, so that

$$c^0 = \sum_{(i,j)\in\mathcal{A}} c_{ij} x_{ij}^0 = \min_{x\in\mathbb{X}} c_{ij} x_{ij} \tag{2}$$

We call the shortest-path represented by x^0 the *perfectly-rational* shortest-path.

On the other hand, a driver with *bounded rationality* may fail to either optimize, have full information, or both. Instead, boundedly rational drivers choose a path if its length is short enough compared to a reference length, if it is perceived as the shortest path based on the limited information of the driver, or if its length is short enough based on the limited information. Boundedly rational drivers satisfice instead of optimize, and have perception error about the network environment, hence limited information instead of full information.

To further describe paths that may be chosen by boundedly rational drivers, we first introduce two definitions for the first source—satisficing behavior—of boundedness in rationality:

Definition 1 (Additive Satisficing). A path is called an *additive satisficing (A-Sat) path*, if the path can be represented by a vector $x \in \mathbb{X}$ such that

$$\text{(A-Sat)} \qquad \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} \le c^0 + E \tag{3}$$

where E is a nonnegative constant for the additive indifference band.

Definition 2 (Multiplicative Satisficing). A path is called a *multiplicative satisficing (M-Sat) path*, if the path can be represented by a vector $x \in \mathbb{X}$ such that

$$\text{(M-Sat)} \qquad \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} \le (1+\kappa)c^0 \tag{4}$$

where $\kappa \in [0,1)$ is a constant for the multiplicative indifference band.

We then consider the second source—perception error—separately. For modeling a driver who optimizes under perception error, we propose the following problem of x

$$\text{(PE)} \qquad \min_{x\in\mathbb{X}} \sum_{(i,j)\in\mathcal{A}} (c_{ij} - \varepsilon_{ij})x_{ij} \tag{5}$$

for some constant cost vector $\varepsilon \in \mathcal{E}$. The vector ε denotes the network user's *perception error* of link cost, and the set $\mathcal{E}$ is the *set of uncertain perception error*. We will refer to problem (5) as the *perception error* model. When we predict the route-choice behavior of network users, we assume that their perception error is restricted to an uncertainty set $\mathcal{E}$. Note that the sign of ε_{ij} in (5) can be either positive or negative, although we will first focus on nonnegative values then extend our discussion for general cases including negative values later.

In the satisficing models, a key underlying assumption is that network users recognize the link costs precisely, but they determine their routes within indifference bands, i.e., they are satisficing and their decision-making is sub-optimal. However, in the above perception error model, we assume that network users are optimizing in route-choice—seeking the shortest path—based upon their *perceived* link cost $c_{ij} - \varepsilon_{ij}$ for some $\varepsilon \in \mathcal{E}_A$.

The perception error model (5) can be used to produce many different behavioral models of network users depending on the definition of the set $\mathcal{E}$. Instances of $\mathcal{E}$ are

$$\mathcal{E}_A = \left\{ \varepsilon : \sum_{(i,j)\in\mathcal{A}} \varepsilon_{ij} \le E, \quad \varepsilon_{ij} \ge 0 \ \forall (i,j) \in \mathcal{A} \right\} \tag{6}$$

$$\mathcal{E}_M = \left\{ \varepsilon : \sum_{(i,j)\in\mathcal{A}} \varepsilon_{ij} \le \kappa c^0, \quad \varepsilon_{ij} \ge 0 \ \forall (i,j) \in \mathcal{A} \right\} \tag{7}$$

$$\mathcal{E}_L = \left\{ \varepsilon : 0 \le \varepsilon_{ij} \le \frac{\kappa}{1+\kappa} c_{ij} \ \forall (i,j) \in \mathcal{A} \right\} \tag{8}$$

for positive constants E and κ. We will show that the PE model (5) with $\mathcal{E}_A$ and $\mathcal{E}_M$ are equivalent to the A-Sat and M-Sat models, respectively, and $\mathcal{E}_L$ is related to the M-Sat model. With the freedom of choice on the set $\mathcal{E}$, the perception error model (5) provides a notion of 'generalized' bounded rationality.

2.1. Additive Satisficing with $\mathcal{E}_A$

We first consider $\mathcal{E}_A$ and show that the perception error model (5) is equivalent to additive satisficing.

THEOREM 1 (PE $+\,\mathcal{E}_A \implies$ A-Sat). *Let $\bar{x}$ be an optimal solution to* (5) *for some $\bar{\varepsilon} \in \mathcal{E}_A$. Then $\bar{x}$ represents an A-Sat path with indifference band E, i.e., $\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} \le c^0 + E$.*

Proof of Theorem 1. We adopt the idea of Lou et al. (2010, Theorem 2.1). Since $\bar{x}$ is an optimal solution to (5) for some $\bar{\varepsilon} \in \mathcal{E}_A$ and $\bar{\varepsilon} \ge 0$, we have

$$\sum_{(i,j)\in\mathcal{A}} (c_{ij} - \bar{\varepsilon}_{ij})\bar{x}_{ij} = \min_{x\in\mathbb{X}} (c_{ij} - \bar{\varepsilon}_{ij}) x_{ij} \le \min_{x\in\mathbb{X}} c_{ij} x_{ij} = c^0.$$

Therefore,

$$\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} - \sum_{(i,j)\in\mathcal{A}} \bar{\varepsilon}_{ij}\bar{x}_{ij} \le c^0. \tag{9}$$

Since $\sum_{(i,j)\in\mathcal{A}} \bar{\varepsilon}_{ij} \le E$, we have $\sum_{(i,j)\in\mathcal{A}} \bar{\varepsilon}_{ij}\bar{x}_{ij} \le E$. Using (9), we obtain

$$\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} \le c^0 + \sum_{(i,j)\in\mathcal{A}} \bar{\varepsilon}_{ij}\bar{x}_{ij} \le c^0 + E,$$

which completes the proof. $\square$

Theorem 1 provides a sufficient condition for a certain path to be an A-Sat path. We can also show that the same conditions provide necessary conditions. The procedure investigates the existence of a solution to an inverse optimization problem. Motivated by the road-pricing literature (Yang and Huang 2004, Marcotte et al. 2009), we first consider the following auxiliary problem:

$$\min_{x,u} \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} + Eu \tag{10}$$

subject to

$$-\sum_{(i,j)\in\mathcal{A}} x_{ij} + \sum_{(j,i)\in\mathcal{A}} x_{ji} = -b_i \quad \forall i \in \mathcal{N} \qquad (\pi_i) \tag{11}$$

$$x_{ij} \geq \bar{x}_{ij} - u \quad \forall (i,j) \in \mathcal{A} \qquad (\lambda_{ij}) \tag{12}$$

$$x_{ij} \geq 0 \quad \forall (i,j) \in \mathcal{A} \tag{13}$$

$$u \geq 0 \tag{14}$$

where π_i and λ_{ij} are dual variables.

Lemma 1. *Suppose $\bar{x} \in \mathbb{X}$ is an A-Sat path with indifference band E such that $\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} \leq \sum_{(i,j)\in\mathcal{A}} c_{ij}x^0_{ij} + E$. Then $\bar{x}$ with $u = 0$ is an optimal solution to the auxiliary problem* (10).

Proof of Lemma 1. First observe that an optimal u is no greater than 1. Then we consider a change of variables:

$$x_{ij} = (1-u)\bar{x}_{ij} + y_{ij}$$

for all $(i,j) \in \mathcal{A}$. Then constraints (12) and (13) require

$$y_{ij} \geq u\bar{x}_{ij} - u,$$

$$y_{ij} \geq -(1-u)\bar{x}_{ij},$$

which leads to $y_{ij} \geq 0$ given that $\bar{x}_{ij}$ is either 0 or 1.

Therefore we can rewrite the auxiliary problem (10) as follows:

$$\min_{y,u} w(y,u) = (1-u) \sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} + \sum_{(i,j)\in\mathcal{A}} c_{ij}y_{ij} + Eu$$

subject to

$$-\sum_{(i,j)\in\mathcal{A}} y_{ij} + \sum_{(j,i)\in\mathcal{A}} y_{ji} = -ub_i \quad \forall i\in\mathcal{N}$$

$$y_{ij} \geq 0 \quad \forall (i,j)\in\mathcal{A}$$

$$u \geq 0$$

When $u=0$, we observe that $y=0$ and $w(0,0)=\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij}$. For any feasible pair (y,u), we have the following bounds on $w(y,u)$:

$$\begin{aligned} w(y,u) &\geq (1-u)\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} + u\sum_{(i,j)\in\mathcal{A}} c_{ij}x^0_{ij} + Eu \\ &\geq (1-u)\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} + u\left(\sum_{(i,j)\in\mathcal{A}} c_{ij}x^0_{ij} + E\right) \\ &\geq (1-u)\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} + u\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} \\ &= \sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} \\ &= w(0,0) \end{aligned}$$

Therefore, $u=0$, $y=0$, and the corresponding flow $x=\bar{x}$ provides an optimal solution to the auxiliary problem (10). This completes the proof. □

The above auxiliary problem (10) plays a key role to derive the necessary conditions. We use the existence of dual variables π and λ of the auxiliary problem (10) to prove the following result and that the dual variable λ is equal to the perception error vector ε.

THEOREM 2 ($\mathsf{PE}+\mathcal{E}_A \Longleftarrow \mathsf{A\text{-}Sat}$). *Suppose $\bar{x}\in\mathbb{X}$ is an A-Sat path within indifference band E. Then there exists a vector $\varepsilon\in\mathcal{E}_A$ such that $\bar{x}$ is an optimal solution to*

$$\min_{x\in\mathbb{X}} \sum_{(i,j)\in\mathcal{A}} (c_{ij}-\varepsilon_{ij})x_{ij} \tag{15}$$

Proof of Theorem 2. The optimality conditions for the auxiliary problem (10) are:

$$x_{ij}(c_{ij}+\pi_i-\pi_j-\lambda_{ij}) = 0 \qquad \forall (i,j)\in\mathcal{A} \tag{16}$$

$$c_{ij} + \pi_i - \pi_j - \lambda_{ij} \geq 0 \qquad \forall (i,j) \in \mathcal{A} \tag{17}$$

$$(x_{ij} + u - \bar{x}_{ij})\lambda_{ij} = 0 \qquad \forall (i,j) \in \mathcal{A} \tag{18}$$

$$u\big(E - \sum \lambda_{ij}\big) = 0 \qquad \forall (i,j) \in \mathcal{A} \tag{19}$$

$$E - \sum \lambda_{ij} \geq 0 \tag{20}$$

$$\lambda_{ij} \geq 0 \qquad \forall (i,j) \in \mathcal{A}. \tag{21}$$

From Lemma 1, we know the pair of vectors $u = 0$ and $x = \bar{x}$ is an optimal solution. Then constraints (18) and (19) hold. Constraints (20) and (21) are equivalent to $\lambda \in \mathcal{E}_A$.

On the other hand, after relaxing $\mathbb{X}$ to X, defined

$$X = \left\{ x : \sum_{(i,j)\in\mathcal{A}} x_{ij} - \sum_{(j,i)\in\mathcal{A}} x_{ji} = b_i \ \forall i \in \mathcal{N}, \quad x_{ij} \geq 0 \ \forall (i,j) \in \mathcal{A} \right\},$$

the optimality conditions for problem (15) are:

$$x_{ij}(c_{ij} - \varepsilon_{ij} + \mu_i - \mu_j) = 0 \qquad \forall (i,j) \in \mathcal{A} \tag{22}$$

$$c_{ij} - \varepsilon_{ij} + \mu_i - \mu_j \geq 0 \qquad \forall (i,j) \in \mathcal{A} \tag{23}$$

$$x \in X \tag{24}$$

Thus using (16), (17), (20) and (21), we can show that there exist vectors π and ε such that (22)–(24) hold; in particular $x = \bar{x}$, $\mu = \pi$, and $\varepsilon = \lambda$. Note that $\bar{x} \in X$, since $\bar{x} \in \mathbb{X}$. $\square$

Note that Definition 1 for additive satisficing allows cycles in paths. In the perception-error model (5) with $\mathcal{E}_A$, it is also possible to generate paths with cycles by making $c_{ij} - \varepsilon_{ij} = 0$. When it is desirable to avoid cyclic paths in the perception-error model, one must add certain cycle-elimination constraints (see, for example, Taccari 2016); then, of course, the exact equivalence, particularly Theorem 2, no longer holds.

For the traffic equilibrium problem with satisficing drivers, called the boundedly rational user equilibrium (BRUE) problem in the literature, Lou et al. (2010) provide link-based conditions similar to the optimality conditions of (15) that produce only acyclic paths. Di et al. (2013) provide an equivalence for certain indifference function between BRUE and a nonlinear complementarity problem (NCP). These two models discussed are both in the setting of equilibrium analysis considering congestion. The link-based model by Lou et al. (2010) produces only a subset of the paths defined in the original path-based bounded rationality. The NCP proposed by Di et al. (2013) obtains a BRUE solution based on a path-based formulation. Note that our the perception-error model is link-based and provides the equivalence between additive satisfying paths and a minimization problem in congestion-free networks.

We will observe a similar relationship between the perception-error model and the multiplicative satisficing models.

2.2. Multiplicative Satisficing with $\mathcal{E}_M$

Noting that $\mathcal{E}_M$ is identical to $\mathcal{E}_A$ with $E = \kappa c^0$, we immediately obtain the following results:

THEOREM 3 ($\mathsf{PE} + \mathcal{E}_M \implies \mathsf{M\text{-}Sat}$). *Let $\bar{x}$ be an optimal solution to* (5) *for some $\bar{\varepsilon} \in \mathcal{E}_M$. Then $\bar{x}$ represents an M-Sat path with indifference band κ, i.e., $\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} \leq (1+\kappa)c^0$.*

THEOREM 4 ($\mathsf{PE} + \mathcal{E}_M \impliedby \mathsf{M\text{-}Sat}$). *Suppose $\bar{x} \in \mathbb{X}$ is an M-Sat path with indifference band κ. Then there exists a vector $\varepsilon \in \mathcal{E}_M$ such that $\bar{x}$ is an optimal solution to*

$$\min_{x\in\mathbb{X}} \sum_{(i,j)\in\mathcal{A}} (c_{ij} - \varepsilon_{ij})x_{ij}. \tag{25}$$

The additive and multiplicative satisficing models can exhibit significantly different behavior, when they are applied to a network design problem in which the network operator can open or close certain links. The *available* shortest path can change depending on the network design variables; then the meaning of additive and multiplicative indifference bands can vary within the network design problem. While the additive indifference band E is always constant, the multiplicative indifference band κc^0 will change according to the length of the available shortest path c^0.

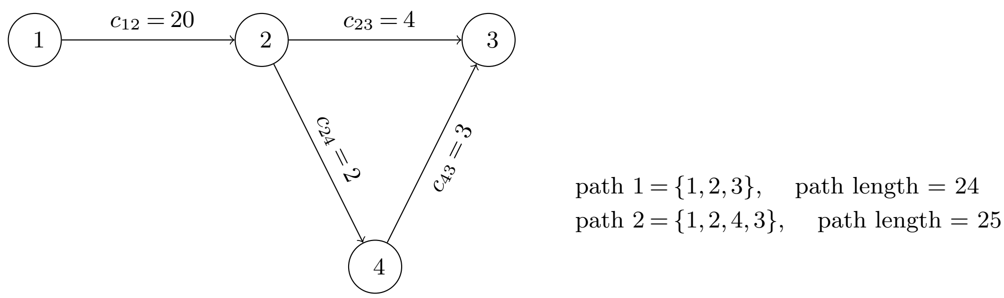


Figure 1 An example where the PE model with $\mathcal{E}_L$ cannot produce a particular M-Sat path. When $\kappa = 0.1$, path 2 is an M-Sat path, but is not a solution to the PE model for any $\varepsilon \in \mathcal{E}_L$.

2.3. Subpath Multiplicative Satisficing with $\mathcal{E}_L$

We will observe that $\mathcal{E}_L$ is another perception-error set that is related to the multiplicative indifference band. In $\mathcal{E}_L$, the perception error is specified for each link instead of restrictions on its sum: $0 \leq \varepsilon_{ij} \leq \frac{\kappa}{1+\kappa} c_{ij}$. We first note that not all M-Sat paths can be produced by the PE model with $\mathcal{E}_L$. Figure 1 illustrates an example. From node 1 to node 3, path 1 is the perfectly rational shortest path with length 24. With $p = 0.1$, path 2 is an M-Sat path with length 25 which is less than $(1 + 0.1)24 = 26.4$. Let us now try to make path 2 a shortest path in the PE model using a perception-error vector $\varepsilon \in \mathcal{E}_L$. First note that increasing ε_{12} will not help, since it will also reduce the length of path 1. The only possibility is when we increase ε_{24} and ε_{43} as much as we can. That is, we set $\varepsilon_{24} = \frac{\kappa}{1+\kappa} c_{24} = 0.182$ and similarly $\varepsilon_{43} = 0.273$. Then the perceived length of path 2 will become 24.545, which is still greater than the length of path 1; i.e., path 2 cannot be produced by the PE model with $\mathcal{E}_L$.

We now introduce the notion of subpath multiplicative satisficing.

Definition 3 (Subpath Multiplicative Satisficing). A path is called a *subpath multiplicative satisficing (SM-Sat) path* with a constant κ, if every subpath of the path is an M-Sat path with the same constant κ between the corresponding origin and destination nodes.

The above definition of 'subpath satisficing' is analogous to the 'subpath optimality' property of the shortest path. We can interpret the underlying behavioral assumption in Definition 3 as follows. After determining the route to take at the origin, drivers may re-evaluate their route to the destination. This interpretation is also related to the sequential selection process modeled in the Markovian traffic equilibrium (Baillon and Cominetti 2008) and the link-based logit model (Fosgerau et al. 2013), both based on RUM, wherein drivers make a choice of links at each node with the probabilities described by RUM.

For example, drivers may like to stop by some mid-points on the way, for example, to visit a gas station, a coffee house, or a grocery store, or to pick someone up. Drivers may also re-evaluate while they are stopping at a traffic signal and decide to change their route. When they resume their trip, we assume that they again consider the multiplicative indifference band, while we are uncertain about their reasoning. In the example in Figure 1, if a driver would have stopped at node 2, he or she would not consider subpath $\{2,4,3\}$ since it is not an M-Sat path from node 2 to node 4.

Mathematically, Definition 3 can be written as follows. Consider a path $\bar{p}$ that is an ordered set of nodes and a flow vector $\bar{x} \in \mathbb{X}$ that represents path $\bar{p}$. To consider a subpath, pick a pair of nodes r and s from $\bar{p}$ such that $\mathrm{ord}_{\bar{p}}(r) < \mathrm{ord}_{\bar{p}}(s)$, where $\mathrm{ord}_{\bar{p}}(r)$ denotes the order of node r in path $\bar{p}$. That is, the flow is directed from r to s. We define the set of path vectors from r to s:

$$\mathbb{X}_{rs} = \left\{ x : \sum_{(i,j)\in\mathcal{A}} x_{ij} - \sum_{(j,i)\in\mathcal{A}} x_{ji} = b_i^{rs} \ \forall i \in \mathcal{N}, \quad x_{ij} \in \{0,1\} \ \forall (i,j) \in \mathcal{A} \right\}$$

where b_i^{rs} has value 1 if $i = r$, -1 if $i = s$, and 0 otherwise. We let $\bar{x}^{rs} \in \mathbb{X}_{rs}$ be a flow vector from r to s along $\bar{x}$. Then $\bar{p}$ is an M-Sat path if

$$\sum_{(i,j)\in\mathcal{A}} c_{ij} \bar{x}_{ij}^{rs} \le (1+\kappa) \min_{x^{rs}\in\mathbb{X}_{rs}} \sum_{(i,j)\in\mathcal{A}} c_{ij} x_{ij}^{rs} \tag{26}$$

for all (r,s) pairs for subpaths. By letting $r = o$ and $s = d$, we observe that an SM-Sat path is also an M-Sat path. Hence, the set of SM-Sat paths is a subset of the set of M-Sat paths.

We first show that the PE model with $\mathcal{E}_L$ produces an SM-Sat path.

THEOREM 5 ($\mathsf{PE} + \mathcal{E}_L \implies \mathsf{SM\text{-}Sat} \implies \mathsf{M\text{-}Sat}$). *Let $\bar{x}$ be an optimal solution to* (5) *for some $\bar{\varepsilon} \in \mathcal{E}_L$. Then $\bar{x}$ represents an SM-Sat—hence also M-Sat—path.*

Proof of Theorem 5. When $\bar{x}$ is an optimal solution to the PE problem (5) for some $\bar{\varepsilon} \in \mathcal{E}_L$, it is well known that any subpath $\bar{x}^{rs}$ is also optimal for (r,s) with the same $\bar{\varepsilon}$ (Cormen et al. 2009, Lemma 24.1). Therefore, we have

$$\sum_{(i,j)\in\mathcal{A}} (c_{ij} - \bar{\varepsilon}_{ij})\bar{x}^{rs}_{ij} = \min_{x^{rs}\in\mathbb{X}_{rs}} (c_{ij} - \bar{\varepsilon}_{ij})x^{rs}_{ij} \le \min_{x^{rs}\in\mathbb{X}_{rs}} c_{ij}x^{rs}_{ij}. \tag{27}$$

Since $0 \le \bar{\varepsilon}_{ij} \le \frac{\kappa}{1+\kappa}c_{ij}$, we have

$$\sum_{(i,j)\in\mathcal{A}} \left(c_{ij} - \frac{\kappa}{1+\kappa}c_{ij}\right)\bar{x}^{rs}_{ij} \le \sum_{(i,j)\in\mathcal{A}} (c_{ij} - \varepsilon_{ij})\bar{x}^{rs}_{ij}. \tag{28}$$

Based on (27) and (28), we obtain

$$\sum_{(i,j)\in\mathcal{A}} \left(c_{ij} - \frac{\kappa}{1+\kappa}c_{ij}\right)\bar{x}^{rs}_{ij} \le \min_{x^{rs}\in\mathbb{X}_{rs}} c_{ij}x^{rs}_{ij},$$

which is identical to (26). Since the choice of (r,s) was arbitrary, this proves the theorem. $\square$

While an M-Sat path may not be represented by a solution to the PE model with $\mathcal{E}_L$, an SM-Sat path can always be, as proved in the following theorem.

THEOREM 6 (PE $+\,\mathcal{E}_L \Longleftarrow$ SM-Sat). *Suppose $\bar{x} \in \mathbb{X}$ is an SM-Sat path with multiplicative indifference band κ. Then there exists a vector $\varepsilon \in \mathcal{E}_L$ so that $\bar{x}$ is an optimal solution to*

$$\min_{x\in\mathbb{X}} \sum_{(i,j)\in\mathcal{A}} (c_{ij} - \varepsilon_{ij})x_{ij} \tag{29}$$

In particular, the following $\bar{\varepsilon}$ is such an $\varepsilon \in \mathcal{E}_L$:

$$\bar{\varepsilon}_{ij} = \begin{cases} \frac{\kappa}{1+\kappa}c_{ij} & \text{if } \bar{x}_{ij} = 1 \\ 0 & \text{if } \bar{x}_{ij} = 0. \end{cases} \tag{30}$$

Proof of Theorem 6. Let us first consider the cases when $\bar{x}$ has *no common links* with another path $\widehat{x} \in \mathbb{X}$. We obtain

$$\begin{aligned} \sum_{(i,j)\in\mathcal{A}} (c_{ij} - \bar{\varepsilon}_{ij})\bar{x}_{ij} &= \sum_{(i,j)\in\mathcal{A}} \left(c_{ij} - \frac{\kappa}{1+\kappa}c_{ij}\right)\bar{x}_{ij} \\ &= \frac{1}{1+\kappa}\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} \\ &\le \sum_{(i,j)\in\mathcal{A}} c_{ij}\widehat{x}_{ij} \\ &= \sum_{(i,j)\in\mathcal{A}} (c_{ij} - \bar{\varepsilon}_{ij})\widehat{x}_{ij} \end{aligned} \tag{31}$$

where we used the property of M-Sat paths. Therefore, $\bar{x}$ is a better solution to the PE problem (29) than $\widehat{x}$.

Now suppose $\bar{x}$ has *some common links* with another path $\widehat{x}$. The two paths will share some subpaths and will have some own subpaths. Suppose there is only one unique subpath for each of $\bar{x}$ and $\widehat{x}$. Let us define:

$$\overline{\mathcal{A}} = \{(i,j) \in \mathcal{A} : \bar{x}_{ij} = 1\}$$

$$\widehat{\mathcal{A}} = \{(i,j) \in \mathcal{A} : \widehat{x}_{ij} = 1\}$$

Links in the unique subpaths of $\bar{x}$ and $\widehat{x}$ are $\overline{\mathcal{A}} \setminus \widehat{\mathcal{A}}$ and $\widehat{\mathcal{A}} \setminus \overline{\mathcal{A}}$, respectively. Note $\bar{\varepsilon}_{ij} = 0$ for all $(i,j) \in \widehat{\mathcal{A}} \setminus \overline{\mathcal{A}}$. We have

$$\begin{aligned}
\sum_{(i,j)\in\mathcal{A}} (c_{ij} - \bar{\varepsilon}_{ij})\bar{x}_{ij} &= \sum_{(i,j)\in\overline{\mathcal{A}}} (c_{ij} - \bar{\varepsilon}_{ij}) \\
&= \sum_{(i,j)\in\overline{\mathcal{A}}\cap\widehat{\mathcal{A}}} (c_{ij} - \bar{\varepsilon}_{ij}) + \sum_{(i,j)\in\overline{\mathcal{A}}\setminus\widehat{\mathcal{A}}} (c_{ij} - \bar{\varepsilon}_{ij}) \\
&\le \sum_{(i,j)\in\overline{\mathcal{A}}\cap\widehat{\mathcal{A}}} (c_{ij} - \bar{\varepsilon}_{ij}) + \sum_{(i,j)\in\widehat{\mathcal{A}}\setminus\overline{\mathcal{A}}} (c_{ij} - \bar{\varepsilon}_{ij}) \\
&= \sum_{(i,j)\in\mathcal{A}} (c_{ij} - \bar{\varepsilon}_{ij})\widehat{x}_{ij}
\end{aligned}$$

where we used (31) for comparing perceived lengths of the subpaths. Therefore $\bar{x}$ is a better solution than $\widehat{x}$. When there is more than one unique subpath, we can repeatedly apply this procedure to show that $\bar{x}$ is a better solution to the PE problem (29) than any other flow vector. □

We have observed so far that the PE model with $\mathcal{E}_L$ is equivalent with the SM-Sat assumption, which is a subset of the set of all M-Sat paths. Path 2 in Figure 1 is an M-Sat path, but not an SM-Sat path.

Furthermore, we observe the following:

LEMMA 2. *Consider a flow vector $\bar{x} \in \mathbb{X}$. The following statements are equivalent:*

(i) A path represented by $\bar{x}$ is an SM-Sat path.

(ii) $\bar{x}$ is an optimal solution to (5) *for $\bar{\varepsilon}$ in* (30).

(iii) $\bar{x}$ *is an optimal solution to* (5) *for some* $\varepsilon \in \mathcal{E}_L$.

Proof of Lemma 2. We observe that *(i)* $\Longrightarrow$ *(ii)* by Theorem 6; *(ii)* $\Longrightarrow$ *(iii)* by definition; and *(iii)* $\Longrightarrow$ *(i)* by Theorem 5. □

Lemma 2, particularly *(i)* $\Longleftrightarrow$ *(ii)*, provides an easy way to check if any given flow vector $\bar{x}$ represents an SM-Sat path or not. After constructing $\bar{\varepsilon}$ as in (30), we use *(ii)* as a test. With $\bar{\varepsilon}$, we solve (5), and compare the optimal objective function value with the objective function value at $\bar{x}$. If $\bar{x}$ is not an optimal solution to (5) with $\bar{\varepsilon}$, then $\bar{x}$ does not represent an SM-Sat path. If it is an optimal solution, then it represents an SM-Sat path.

We also note that an SM-Sat path, or equivalently a path by PE with $\mathcal{E}_L$, cannot have a cycle, unless there exists a subpath with zero travel cost. This is a unique feature of SM-Sat paths, compared to A-Sat and M-Sat paths.

3. Generalized Bounded Rationality

As we have observed, for the additive and multiplicative satisficing cases, we can use the general framework (5) with $\mathcal{E}_A$ and $\mathcal{E}_M$, respectively. The notion of 'perception error' is more general than additive and multiplicative satisficing, and we can consider various sets of uncertain perception errors $\mathcal{E}$. For example, one may consider:

$$\mathcal{E}_H = \left\{ \varepsilon : \sum_{(i,j)\in\mathcal{A}} \varepsilon_{ij} \le E, \sum_{(i,j)\in\mathcal{A}} \varepsilon_{ij} \le \kappa c^0 \right\} \tag{32}$$

$$\mathcal{E}_B = \left\{ \varepsilon : l_{ij} \le \varepsilon_{ij} \le u_{ij} \quad \forall (i,j) \in \mathcal{A} \right\} \tag{33}$$

$$\mathcal{E}_E = \left\{ \varepsilon : ||Q^{-1/2}\varepsilon||_2 \le \xi \right\} \tag{34}$$

To provide a hybrid definition of additive and multiplicative satisficing, one may consider $\mathcal{E}_H$. It will generate paths whose lengths are less than $\min\{c^0 + E, (1+\kappa)c^0\}$. Certainly, $\mathcal{E}_L$ is a special case of the box set $\mathcal{E}_B$. With the choice of $\mathcal{E}_B$ when l_{ij} can be negative, we assume that the perception of network users can not only decrease link costs but also increase. We note that $\mathcal{E}_H \cap \mathcal{E}_B$ is related to the polyhedral uncertainty set considered in the robust optimization literature (Bertsimas and Sim 2003, 2004, Kwon et al. 2013).

In the ellipsoidal set $\mathcal{E}_E$, note that Q is the covariance matrix of link perception errors and its size is $|\mathcal{A}| \times |\mathcal{A}|$. We assume that Q is symmetric and positive semi-definite, and ξ is a positive constant. The ellipsoidal set $\mathcal{E}_E$ is also popularly considered in the robust optimization literature (Ben-Tal and Nemirovski 1998, 2002, El Ghaoui et al. 1998). Other possibilities of $\mathcal{E}$ found in the robust optimization literature are cardinality constrained sets and norm sets (Bertsimas et al. 2011).

In the following theorem, we show that the perception-error model with the ellipsoidal set $\mathcal{E}_E$ provides a certain bound on the rationality of network users.

THEOREM 7. *Let $\bar{x}$ be an optimal solution to (5) for some $\varepsilon \in \mathcal{E}_E$. Then,*

$$\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} \le c^0 + \xi\sqrt{\bar{x}^T Q \bar{x}} \tag{35}$$

Furthermore, the following bound holds

$$\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} \le c^0 + \frac{\xi^2}{2} + \xi\sqrt{c^0 + \frac{\xi^2}{4}} \tag{36}$$

in a special case when $Q = \mathrm{diag}(\ldots, c_{ij}, \ldots)$.

Proof of Theorem 7. From the proof of Theorem 1, we know that

$$\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} - \sum_{(i,j)\in\mathcal{A}} \varepsilon_{ij}\bar{x}_{ij} \le c^0.$$

Since $\max_{\varepsilon\in\mathcal{E}_E} \varepsilon^T \bar{x} = \xi\sqrt{\bar{x}^T Q \bar{x}}$, we obtain

$$c^0 \ge \sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} - \sum_{(i,j)\in\mathcal{A}} \varepsilon_{ij}\bar{x}_{ij} \ge \sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} - \xi\sqrt{\bar{x}^T Q \bar{x}}$$

When $Q = \mathrm{diag}(\ldots, c_{ij}, \ldots)$, we have

$$\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} \le c^0 + \xi\sqrt{\sum_{(i,j)\in\mathcal{A}} c_{ij}(\bar{x}_{ij})^2} = c^0 + \xi\sqrt{\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij}}$$

since $\bar{x}_{ij}$ is binary. Let $\sigma = \sqrt{\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij}}$, then

$$\begin{aligned}
&\sigma^2 - \xi\sigma \le c^0 \\
&\Longrightarrow (\sigma - \xi/2)^2 \le c^0 + \xi^2/4 \\
&\Longrightarrow \sigma \le \sqrt{c^0 + \xi^2/4} + \xi/2 \\
&\Longrightarrow \sigma^2 \le c^0 + \xi^2/2 + \xi\sqrt{c^0 + \xi^2/4}
\end{aligned}$$

Therefore, we obtain the theorem. □

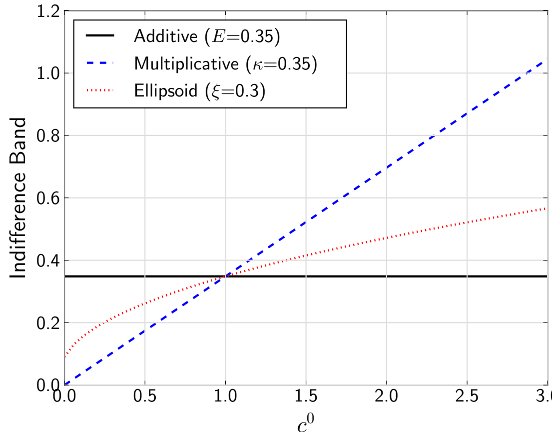


Figure 2 Comparing the indifference bands among the cases of $\mathcal{E}_A$, $\mathcal{E}_M$, and $\mathcal{E}_E$. The parameter values E, κ, and ξ are chosen so that the indifference bands are identical at $c^0 = 1$. We consider the special case of $Q = \text{diag}(..., c_{ij}, ...)$.

While interpreting the right-hand sides of (35) and (36) is not as clear as in the additive or multiplicative satisficing cases, they certainly demonstrate that the ellipsoidal set of perception errors leads to some boundedness in the rationality of network users.

In Figure 2, we compare the size of the indifference band. For each of $\mathcal{E}_A$, $\mathcal{E}_M$, and $\mathcal{E}_E$ with $Q = \text{diag}(..., c_{ij}, ...)$, the indifference band is defined as E, κc^0, and $\frac{\xi^2}{2} + \xi\sqrt{c^0 + \frac{\xi^2}{4}}$, respectively. While the size of the indifference band is constant with additive satisficing, it increases with respect to the length of the shortest path with multiplicative satisficing. In the case of ellipsoidal set $\mathcal{E}_E$, the size of the indifference band falls between the additive and multiplicative satisficing cases. If the shortest path is fixed, all three bounded rationalities in Figure 2 can be made equivalent. However, the shortest path can change if the network topology changes, as in network design problems, where the above three bounded rationalities imply distinct modeling of user behavior.

Observations so far motivate us to define the notion of *generalized bounded rationality*:

Definition 4 (Generalized Bounded Rationality). A network user possesses *generalized bounded rationality* if the user's route-choice decision-making can be justified by the perception error model (5) for some closed and bounded set $\mathcal{E}$.

We consider the closedness and boundedness of the set $\mathcal{E}$ to guarantee that a solution to the perception error model exists and the length of the resulting route is bounded.

Definition 4 based on the perception-error model is general in two perspectives, one of which comes obviously from the generality of $\mathcal{E}$ choices. The other comes from the observation that, with some choices of $\mathcal{E}$, the perception-error model becomes equivalent to one of the existing bounded rationality models that assume satisficing drivers. In addition, we can model both sources of boundedness in rationality, namely, satisficing drivers and an individual's perception of decision environments, in the framework of the perception-error model (5). Let us consider a driver who satisfices rather than optimizes, with their own perception on link travel costs. This driver can be modeled by the following optimization problem:

$$\min_{x \in \mathbb{X}} \sum_{(i,j) \in \mathcal{A}} (c_{ij} - \varepsilon_{ij} - \delta_{ij}) x_{ij} \tag{37}$$

for some $\varepsilon \in \mathcal{E}_1$ and $\delta \in \mathcal{E}_2$ where ε and $\mathcal{E}_1$ are related to the driver's perception error, and δ and $\mathcal{E}_2$ are related to the driver's satisficing criterion.

Consideration of generalized bounded rationality in the form of the perception-error model with a choice of the perception-error set $\mathcal{E}$ brings a significant advantage. While the existing notions of bounded rationality are defined on paths, we can have link-based *preference* control in the perception-error model. That is, if network users are known to prefer to use a certain link (i,j) despite its higher link cost, we can enforce a large perception-error ε_{ij}. This opens an opportunity for generalized bounded rationality to be applied to the cases with an available network user survey.

Under a choice of $\mathcal{E}$, one can find boundedly rational shortest paths with the following conditions:

$$x_{ij}(c_{ij} - \varepsilon_{ij} + \pi_i - \pi_j) = 0 \qquad \forall (i,j) \in \mathcal{A} \tag{38}$$

$$c_{ij} - \varepsilon_{ij} + \pi_i - \pi_j \geq 0 \qquad \forall (i,j) \in \mathcal{A} \tag{39}$$

$$x \in \mathbb{X} \tag{40}$$

$$\varepsilon \in \mathcal{E} \tag{41}$$

where conditions (38) and (39) are the complementary slackness and dual feasibility conditions of the LP form of problem (5), respectively. Alternatively, condition (38) can be replaced by the strong duality condition

$$\sum_{(i,j)\in\mathcal{A}} (c_{ij} - \varepsilon_{ij}) x_{ij} = \pi_d - \pi_o \tag{42}$$

where o and d are the origin and destination nodes, respectively. Note that both approaches involve bi-product terms $\varepsilon_{ij} x_{ij}$.

4. Choosing Indifference Bands

There are parameter values that define the set of perception errors, which we call *indifference bands*; for example, E for $\mathcal{E}_A$ as in additive satisficing, κ for $\mathcal{E}_M$ and $\mathcal{E}_L$ as in (subpath) multiplicative satisficing, and ξ for the ellipsoidal set $\mathcal{E}_E$. One may conduct a survey among drivers to determine an appropriate indifference band value. When a survey report is unavailable, one may use the random utility model (RUM) as a reference. Although it is more desirable to have an independent modeling and data collection approach for the perception error model, benchmarking with widely used RUM parameter estimations would make the transition to a new modeling paradigm easier.

This section compares the bounded rationality models with RUM and estimates the path length distribution by a Monte Carlo method to suggest a method to determine the indifference band values. While this section is written for κ in $\mathcal{E}_M$ and $\mathcal{E}_L$, we can use a similar approach for E in $\mathcal{E}_A$ and ξ in $\mathcal{E}_E$.

4.1. Comparison with the Random Utility Model

For RUM, a random error term is associated with the utility function to capture perception error of travelers and attributes that are unobservable to analysts. The utility of using the k-th path, U_k is given by

$$U_k = -\frac{c_k}{\theta} + \xi_k, \qquad \forall\ k,$$

where c_k is the generalized cost of all the observed attributes, θ is the scale parameter, and ξ_k is a random term. Travel time is usually used for c_k when only one attribute is considered. If the random terms ξ_k are independently and identically distributed Gumbel variates, according to McFadden (1978), the choice probability for path k is

$$P_k = \Pr[k \text{ is chosen from } \mathcal{R}] = \frac{\exp(-c_k/\theta)}{\sum_{l\in\mathcal{R}} \exp(-c_l/\theta)} \ ,$$

where $\mathcal{R}$ is the set of all paths between one OD pair. Note that $P_k > 0$ for any path $k \in \mathcal{R}$, regardless how long it is.

We are interested in what probability a satisficing path is chosen according to RUM. In case of multiplicative satisficing with the indifference band κ, we define the *coverage probability* as follows:

$$P_{\text{cov}}(\kappa) = \sum_{k\in\mathcal{R}} \mathbb{I}[c_k \leq (1+\kappa)c^0] \ P_k = \frac{\sum_{k\in\mathcal{R}} \ \mathbb{I}[c_k \leq (1+\kappa)c^0] \ \exp(-c_k/\theta)}{\sum_{l\in\mathcal{R}} \exp(-c_l/\theta)} \tag{43}$$

where $\mathbb{I}[\cdot]$ is an indicator function whose value is 1 if the specified condition within the brackets is true and zero otherwise.

To use (43) to determine κ, we need two items of information: the value of θ and the fully enumerated set of paths $\mathcal{R}$. If we include paths with cycles, the size of $\mathcal{R}$ can be made arbitrarily large. We restrict our discussion to simple paths only. To determine θ, we need to correlate it to how sensitive it is to the error term. In order for at least a fraction M of the errors not to exceed a certain percentage κ' of the c^0 , θ can be obtained as

$$\theta = \min\left\{ \frac{\kappa' c^0}{\gamma - \ln\left(\ln(\frac{1}{q})\right)} \ , \ \frac{-\kappa' c^0}{\gamma - \ln\left(\ln(\frac{1}{1-q})\right)} \right\}, \tag{44}$$

and $q = \frac{M+1}{2}$ and $\gamma \approx 0.57721566$, which denotes the Euler-Mascheroni constant (Bergomi 2009).

For example, we do a numerical analysis on the Buffalo network (Toumazis and Kwon 2013) of OD pair $(1, 84)$ with the paths generated shown in Figure 3. The Buffalo network has 90 nodes and 149 arcs. By choosing $\kappa' = 0.1$ and $M = 90\%$, we obtain the probability of choosing each path in Figure 4. For the multiplicative satisfying model with $\kappa = 0.15$, we are able to cover 89.0% of the probability of choosing paths using RUM.

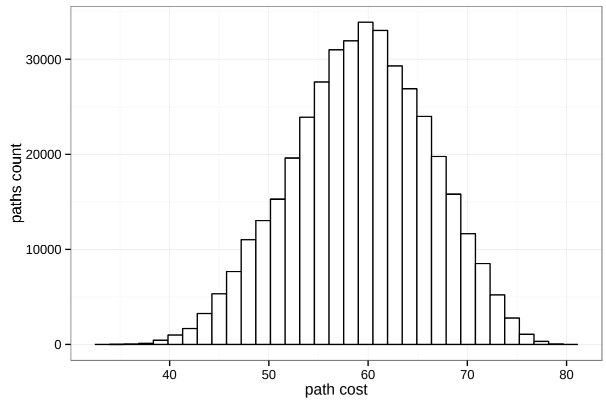


Figure 3 PDF of path set generated on Buffalo network of OD pair $(1,84)$

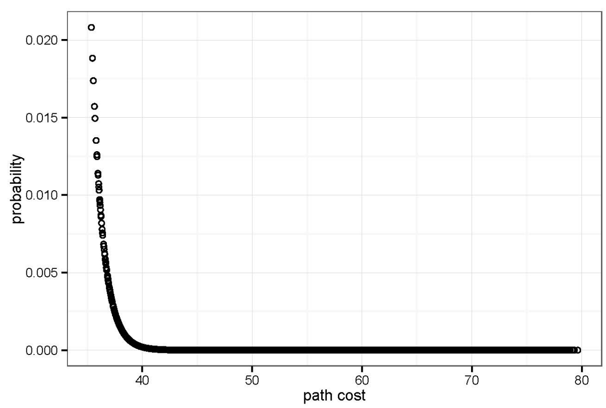


Figure 4 Probability of choosing each path of OD pair $(1,84)$ on Buffalo network

4.2. Monte-Carlo Method for Estimating the Indifference Band

In most realistic road networks, enumerating all paths between an OD pair is a numerically expensive way to compute P_{cov}. Instead, we try to estimate the distribution without full enumeration, using a Monte Carlo method. We restrict our discussion in this section to simple paths. That is, let $\mathcal{R}$ denote the set of all simple paths between the OD pair of interest. We define the set of all simple paths whose length is no greater than $(1+\kappa)c^0$:

$$\mathcal{R}_\kappa = \{l \in \mathcal{R} : c_l \leq (1+\kappa)c^0\},$$

where c_l is the length of path l. We can write

$$P_{\text{cov}}(\kappa) = \frac{N_{\text{cov}}(\kappa)}{D_{\text{cov}}} \tag{45}$$

$$N_{\text{cov}}(\kappa) = \sum_{l \in \mathcal{R}_\kappa} \exp(-c_l/\theta) \tag{46}$$

$$D_{\text{cov}} = \sum_{l \in \mathcal{R}} \exp(-c_l/\theta) \tag{47}$$

where N_{cov} and D_{cov} denote the numerator and denominator, respectively.

We use the *length-distribution method* of Roberts and Kroese (2007) to estimate path cost distribution. Roberts and Kroese (2007) originally used the method to estimate the number of simple paths available between an OD pair. The method involves easy sampling of simple paths and uses a sequential Monte Carlo method to construct an unbiased estimator for the number of available paths. The method randomly generates simple paths that begin at the origin and may or may not end at the destination. By sampling from a larger set of paths, the method estimates the smaller set—the set of paths that ends at the destination of interest.

Let $l^{(1)}, ..., l^{(N)}$ be N sample paths generated by the length-distribution method as described in Roberts and Kroese (2007), with the probability function $g(\cdot)$. When sample path $l^{(i)}$ is indeed a valid path from the origin and the destination, we say $l^{(i)} \in \mathcal{R}$, and when it does not reach the destination, it is invalid and $l^{(i)} \notin \mathcal{R}$. We have $g(l) > 0$ for all $l \in \mathcal{R}$. We can estimate the function value $P_{\text{cov}}(\kappa)$ by the following estimator for any $\kappa \in [0,1]$:

$$\widehat{P_{\text{cov}}(\kappa)} = \frac{\widehat{N_{\text{cov}}(\kappa)}}{\widehat{D_{\text{cov}}}} \tag{48}$$

$$\widehat{N_{\text{cov}}(\kappa)} = \frac{1}{N} \sum_{i=1}^{N} \frac{\mathbb{I}[l^{(i)} \in \mathcal{R}_\kappa] \ \exp(-c_{l^{(i)}}/\theta)}{g(l^{(i)})} \tag{49}$$

$$\widehat{D_{\text{cov}}} = \frac{1}{N} \sum_{i=1}^{N} \frac{\mathbb{I}[l^{(i)} \in \mathcal{R}] \ \exp(-c_{l^{(i)}}/\theta)}{g(l^{(i)})} \tag{50}$$

Given the approximation form of $\widehat{P_{\text{cov}}}(\kappa)$ in (48), we can select κ that guarantees a certain preset level of coverage probability $\overline{P_{\text{cov}}}$. That is, we desire $\widehat{P_{\text{cov}}}(\kappa) \geq \overline{P_{\text{cov}}}$. Since $\widehat{P_{\text{cov}}}(\cdot)$ is a strictly increasing function, we can use a simple line search to find κ such that

$$0 \leq \widehat{P_{\text{cov}}}(\kappa) - \overline{P_{\text{cov}}} < \eta$$

for sufficiently small $\eta > 0$.

Table 1 Comparing the true and estimated κ values for OD pair $(1,84)$ of Buffalo network

$\overline{P_{\text{cov}}}$	True	Estimated	Difference
95%	0.18603	0.18830	1.22%
90%	0.15437	0.15437	0.00%
85%	0.13656	0.13458	1.45%
80%	0.12327	0.12044	2.29%
75%	0.11196	0.10927	2.40%
70%	0.10178	0.09853	3.19%

We sampled $20,000$ paths on the Buffalo network for OD pair $(1,84)$. The results are recorded in Table 1. We can see that the bounds we estimated are close to the real values, with the largest difference as 3.19%. More numerical tests and some insights in SM-Sat paths are provided in Electronic Companion EC.1.

5. Robust Multi-Commodity Network Design Problems

In this section, we formulate and discuss how to incorporate generalized bounded rationality into robust network design problems. When designing a certain network, due to the uncertainty caused by the perception errors of the travelers, it is beneficial to use a robust approach.

For the single origin-destination (OD) case, application could include the shortest path interdiction problem (Israeli and Wood 2002). We can also generalize to network design problems involving multiple ODs when there is no congestion effect such as the shortest path interdiction problem with multiple ODs and hazardous materials network design problem (Kara and Verter 2004). Here we introduce the robust hazmat network design problem as an example. However it will be similar for other applications. The formulation is given as follows:

$$\min_{y} \ \alpha \sum_{(i,j)\in\mathcal{A}} (1-y_{ij}) + \max_{\varepsilon} \sum_{(i,j)\in\mathcal{A}} \sum_{s\in\mathcal{S}} n_s r_{ijs} x_{ijs} \tag{51}$$

$$y_{ij} \in \{0,1\} \quad \forall (i,j)\in\mathcal{A} \tag{52}$$

$$\varepsilon^s \in \mathcal{E}^s \quad \forall s\in\mathcal{S} \tag{53}$$

$$\min_{x} \sum_{(i,j)\in\mathcal{A}} (c_{ij} - \varepsilon_{ijs}) x_{ijs} \tag{54}$$

$$\text{s.t.} \sum_{(i,j)\in\mathcal{A}} x_{ijs} - \sum_{(j,i)\in\mathcal{A}} x_{jis} = b_i^s \quad \forall i\in\mathcal{N} \tag{55}$$

$$x_{ijs} \le y_{ij} \quad \forall (i,j) \in \mathcal{A}, s \in \mathcal{S} \tag{56}$$

$$x_{ijs} \in \{0,1\} \quad \forall (i,j) \in \mathcal{A}, s \in \mathcal{S} \tag{57}$$

Decision variables y_{ij} denote whether one link is open ($y_{ij} = 1$) or closed ($y_{ij} = 0$). The constraints (54)–(57) model the behaviors of travelers taking the shortest path for shipment $s \in \mathcal{S}$ between certain origin and destination, based on the realization of perception error ε_{ijs} and closed links y_{ij}. Constraints (53) make sure the perception uncertainty is within the chosen bounded rationality set. The perception-error set $\mathcal{E}^s$ is different for various models. But generally for $\mathcal{E}_A$, $\mathcal{E}_M$ and $\mathcal{E}_B$, it can be represented as

$$\sum_{(i,j)\in\mathcal{A}} \varepsilon_{ijs} \le (1+\kappa_s)c^0 \quad \forall s \in \mathcal{S} \tag{58}$$

$$\sum_{(i,j)\in\mathcal{A}} \varepsilon_{ijs} \le c^0 + E_s \quad \forall s \in \mathcal{S} \tag{59}$$

$$l_{ijs} \le \varepsilon_{ijs} \le u_{ijs} \quad \forall (i,j) \in \mathcal{A}, s \in \mathcal{S} \tag{60}$$

$$c^0 = \min_z \sum_{(i,j)\in\mathcal{A}} c_{ij} z_{ijs} \tag{61}$$

$$\text{s.t.} \sum_{(i,j)\in\mathcal{A}} z_{ijs} - \sum_{(j,i)\in\mathcal{A}} z_{jis} = b_i^s \quad \forall i \in \mathcal{N} \tag{62}$$

$$z_{ijs} \le y_{ij} \quad \forall (i,j) \in \mathcal{A}, s \in \mathcal{S} \tag{63}$$

$$z_{ijs} \in \{0,1\} \quad \forall (i,j) \in \mathcal{A}, s \in \mathcal{S} \tag{64}$$

where constraints (61)–(64) obtain the shortest path length based on the decision of y_{ij}, and constraints (58)–(60) defines $\mathcal{E}_A$, $\mathcal{E}_M$ and $\mathcal{E}_B$ for each shipment s. For objective (51), it seeks to minimize the worst total risk while considering the number of closed links, similar to the approach in Sun et al. (2016a).

6. Solving Robust Network Design Problems

The robust network design problem is a tri-level mixed integer optimization problem. The lowest level problem, which is a shortest path problem, has the integrality property. Thus we can transform it with its KKT optimality conditions and reduce the robust network design problem to a bilevel integer problem (Labbé et al. 1998, Kara and Verter 2004). However, the problem is still a bilevel integer programming problem, which remains hard to solve. In order to solve the problem, we propose a cutting plane algorithm based on the cuts designed by Gzara (2013). First we define the master problem (MP) and worst case problem (WCP), which will be utilized by the proposed cutting plane algorithm.

6.1. Master Problem

The master problem is the same as the minimum risk network flow problem defined in Gzara (2013). It can be written as follows:

$$
\begin{aligned}
\text{(MP)} \ \min_{y,\varepsilon} \ & \sum_{(i,j)\in\mathcal{A}} \sum_{s\in\mathcal{S}} n_s r_{ijs} x_{ijs} + \alpha \sum_{(i,j)\in\mathcal{A}} (1-y_{ij}), \\
\text{s.t.} \ & y_{ij} \in \{0,1\}, \quad \forall (i,j)\in\mathcal{A}, \\
& \sum_{(i,j)\in\mathcal{A}} x_{ijs} - \sum_{(j,i)\in\mathcal{A}} x_{jis} = b_i^s, \quad \forall i\in\mathcal{N}, \\
& x_{ijs} \le y_{ij}, \quad \forall (i,j)\in\mathcal{A}, s\in\mathcal{S}, \\
& x_{ijs} \in \{0,1\}, \quad \forall (i,j)\in\mathcal{A}, s\in\mathcal{S}, \\
& \{\text{Cuts that are added iteratively}\}.
\end{aligned}
$$

The master problem obtains the minimum risk paths while not violating the cuts that are added iteratively The cuts that are added are supposed to restrict the master problem from taking certain paths, which will be further discussed.

6.2. Worst Case Problem

By solving the master problem, we obtain a solution $\bar{y}$, $\bar{x}$. Then the worst case problem (WCP) can be formulated as:

$$
\text{(WCP)} \ \max_{\varepsilon} \sum_{(i,j)\in\mathcal{A}} r_{ijs} x_{ijs}, \quad \forall s\in\mathcal{S}
$$

$$
\begin{aligned}
\text{s.t. } & \varepsilon^s \in \mathcal{E}^s, \\
& \min_x \sum_{(i,j)\in\mathcal{A}} (c_{ijs} - \varepsilon_{ijs}) x_{ijs}, \\
& \text{s.t.} \quad \sum_{(i,j)\in\mathcal{A}} x_{ijs} - \sum_{(j,i)\in\mathcal{A}} x_{jis} = b_i^s, \quad \forall i \in \mathcal{N}, \\
& \qquad x_{ijs} \le \bar{y}_{ij}, \quad \forall (i,j) \in \mathcal{A}, \\
& \qquad x_{ijs} \in \{0,1\}, \quad \forall (i,j) \in \mathcal{A}.
\end{aligned}
$$

The WCP finds the maximum risk path among all the available paths that could be perceived as shortest under the realization of the perception error set. The lower level problem is a shortest path problem with arc cost as $c_{ijs} - \varepsilon_{ijs}$ for each shipment s; hence we can replace the lower level by its KKT conditions and obtain a single level MILP for the worst case problem.

However, the problem could be ill-defined for sets that have no upper bound enforced on each link such as $\mathcal{E}_M$ or $\mathcal{E}_A$. In these cases, the cost for the link $c_{ijs} - \varepsilon_{ijs}$ can be negative, which makes the problem much harder to solve. The WCP will accept solutions with cycles as optimal. Actually since WCP is a risk maximization problem, it will prefer solutions with cycles. Thus constraints for eliminating cycles (Taccari 2016) should be incorporated in the lower level problem. These constraints will make the lower problem lose the integrality property and the WCP becomes a bilevel mixed integer programming problem.

Instead of solving a bilevel mixed integer programming problem, we could make a simple search on the available paths that could be perceived as shortest. For a set with upper bound on the sum such as $\mathcal{E}_M$ and $\mathcal{E}_A$, we can utilize a K loopless shortest path algorithm to generate the paths that have cost lower than $(1+\kappa)c^0$ and $c^0 + E$ with the realization of $\bar{y}$. After obtaining the path set, we can choose the maximum risk path as the optimal solution of WCP. In case of $\mathcal{E}_E$, a similar procedure may be used. In generating the path set, instead of having a fixed bound, for each path $\bar{x}_{ij}$, we can test whether it has lower cost than $c^0 + \xi\sqrt{\bar{x}^T Q \bar{x}}$ or not.

For $\mathcal{E}_L$, even though we could solve WCP as an MILP, a search algorithm is still applicable and could be faster. We can easily test if a path is an optimal solution to the lower-level problem of

(WCP) using Lemma 2. For $\mathcal{E}_B$, we can similarly show that a path $\bar{p}$ with flow vector $\bar{x}$ is a solution to the lower-level problem for some $\varepsilon \in \mathcal{E}_B$, if and only if $\bar{x}$ is a solution to the lower-level problem for $\bar{\varepsilon} \in \mathcal{E}_L$, where

$$\bar{\varepsilon}_{ij} = \begin{cases} u_{ij} & \text{if } \bar{x}_{ij} = 1, \\ l_{ij} & \text{if } \bar{x}_{ij} = 0. \end{cases}$$

6.3. Cutting Plane Algorithm

After defining the two problems, the cutting plane algorithm to solve the robust network design problem can be summarized as follows:

Step 1 Set the iteration number $k = 1$.

Step 2 Solve the MP and obtain solution $\bar{x}^k$ and $\bar{y}^k$.

Step 3 Using the obtained $\bar{y}^k$ from the MP solve the WCP and obtain solution $\widehat{x}^k$

Step 4 Compare $\widehat{x}^k$ and $\bar{x}^k$. If the two routes solutions are the same, stop. Otherwise, generate valid cuts and add them to the MP. Let $k = k + 1$ and go to *Step 2.*

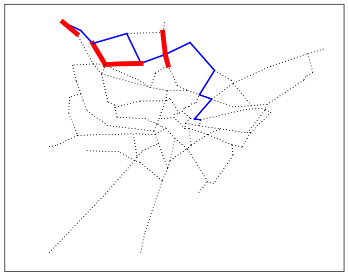
The essence of the algorithm is to compare the minimum risk path and the worst risk path which could be perceived as shortest and make the them the same for both MP and WCP. By adding cuts iteratively, we will be able to change the network design variables to enforce the routes chosen by WCP unavailable for travelers. The WCP obtains a worst path for the first level problem depending on the bounded rationality set. For the algorithm, we use the same cuts as in Gzara (2013), as described in Electronic Companion EC.2.

7. An Application in Hazardous Materials Transportation and Numerical Experiments

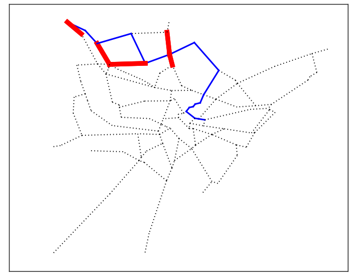
In this section, we show an application of the proposed concept of generalized bounded rationality in the hazmat network design problem (HNDP). The current literature in HNDP assumes carriers always choose the shortest path. However, due to the unobservable attributes of the carriers and lack of knowledge of how they make route decisions, this assumption is questionable. With the notation of generalized bounded rationality, we are able to assign route and link preferences for carriers, which could better capture how carriers make their route decisions. We test the models using the Ravenna (Bonvicini and Spadoni 2008, Erkut and Gzara 2008) network data set transporting four kinds of hazardous materials: chlorine, LPG, gasoline and methanol. The Ravenna network has 105 nodes and 134 undirected arcs. The algorithms are coded in C++ and the CPLEX solver is used. The experiments are run on a high performance computer with 32GB of RAM and a Xeon processor. We illustrate the process of the algorithm in Electronic Companion EC.3 and present the efficiency and computational time of the algorithm in Electronic Companion EC.4.

Now we show the effectiveness of considering generalized bounded rationality in network design problems. We examine cases of HNDP considering M-Sat (HNDP-M-Sat) and HNDP considering SM-Sat (HNDP-SM-Sat). We first compare the results with that of the Deterministic HNDP (DHNDP) on one OD pair (106,71). The results can be found in Figure 5. For the DHNDP, it assumes the carriers choose the shortest path without any perception error, leading to a solution with risk of 761.0 as shown in Figure 5a. However, due to the bounded rationality of the carriers, a set of paths is possible to be chosen. Suppose carriers are SM-Sat with $\kappa = 0.05$, under the decision of DHNDP, another route with higher risk value of 1240.8 (shown in Figure 5b) is probably chosen.

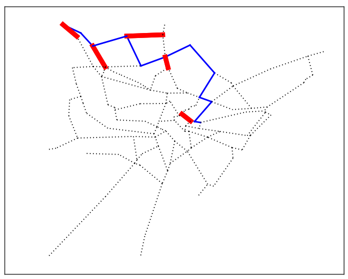
In the cases of considering generalized bounded rationality, we could avoid such worst risk scenarios. We consider a κ value of 0.05. The worst possible risk value is the same with the optimal risk value of the DHNDP. Considering that carriers are SM-Sat, as shown in Figure 5c, we close the same number of road segments as the DHNDP. However, by considering a set of paths which are possible to be chosen by the carriers, we are able to identify a more critical arc to close so that it will block more paths. For the case of HNDP-M-Sat, we close one additional arc compared to HNDP-SM-Sat, even though the resulting route chosen by the carrier is the same, which is $p: \{106, 1, 2, 4, 17, 19, 23, 40, 47, 61, 67, 71\}$ with risk of 767.0 and cost of 28898. As shown in Figure 5d, the additional closed arc is $\{59, 64\}$, and restricts the carrier from choosing path $\widehat{p}: \{106, 1, 2, 4, 17, 19, 23, 40, 59, 64, 61, 67, 71\}$ with risk of 812.0 and cost of 29669. The shortest path without perception error is the path p and clearly the path $\widehat{p}$ cost $29669 < (1 + 0.05) \cdot 28898 = 30342.9$.



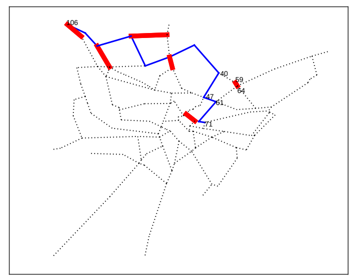
(a) DHNDP network, $R = 767.0$, $C = 28898$



(b) DHNDP worst scenario, $R = 1240.8$, $C = 29099$



(c) HNDP-SM-Sat network, $R = 767.0$, $C = 28898$



(d) HNDP-M-Sat network, $R = 767.0$, $C = 28898$

Figure 5 Solution network comparison of different models

Thus path $\widehat{p}$ is M-Sat and HNDP-M-Sat blocks this path. In the case of HNDP-SM-Sat, path $\widehat{p}$ is evaluated differently. The difference between paths p and $\widehat{p}$ is the supaths $\{40, 47, 61\}$ with cost 4511 and $\{40, 59, 64, 61\}$ with cost $5281 > 4511 \cdot (1 + 0.05) = 4736.6$. Based on the definition of SM-Sat, path $\widehat{p}$ is not SM-Sat. Thus HNDP-SM-Sat need not block path $\widehat{p}$. However, if we are assuming carriers are SM-Sat while they are M-Sat in reality, we could end up with a higher risk of 812.0. But it is still better compared to the worst case scenario of the DHNDP. On the contrary, if we assume carriers are M-Sat and they act as SM-Sat, we achieve the same objective value as planned.

We also make a comparison of the models on 25 test cases we generated as explained in Electronic Companion EC.4. We set $\alpha = 0.025$ and $\kappa = 0.05$. The results are obtained in Table 2. We record the number of closed links and risk values for both HNDP-M-Sat and HNDP-SM-Sat. We find that HNDP-M-Sat always has a larger number of closed links compared to the HNDP-SM-Sat case, closing 3.72 (17.36%) more links on average. HNDP-M-Sat always has higher risk values as well.

Due to the flexibility of closing links, the risk percentage difference is not much, only 0.5%. The reason behind the higher number of closed links and risk values for M-Sat is that SM-Sat is a subset of M-Sat if we use the same κ value. Then HNDP-M-Sat needs to close more links compared to HNDP-SM-Sat. In some cases, it is not efficient to close more links, HNDP-M-Sat would allow more paths to be chosen, leading to higher risk values.

Furthermore, the "M-Sat/D" column in Table 2 demonstrates the risk values of the DHNDP solution assuming M-Sat carrier behaviors and compares that with the HNDP-M-Sat case. Similarly the "SM-Sat/D" column shows the risk values of DHNDP solutions assuming SM-Sat carrier behaviors and compares that with the HNDP-SM-Sat case. Consistent with the results shown in Figure 5, the DHNDP solution would lead to higher risk values. For M-Sat, there is a risk increase of 15.37% on average while only 5.94% for SM-Sat. For the SM-Sat solution, if assuming carriers are M-Sat and could choose more routes, the risk values of DHNDP could be even larger. Thus without considering carriers' bounded rational behaviors, DHNDP could lead to less effective solutions. Additionally, we mark the risk values of HNDP-SM-Sat solution assuming M-Sat carrier behaviors. We observe that there is high risk increase. Thus the HNDP-M-Sat solution is more robust while being more conservative. HNDP-SM-Sat solutions are more effective assuming SM-Sat carrier behavior.

To sum up, by utilizing the notion of generalized bounded rationality in HNDP, we are able to achieve a better worst case scenario, even without any additional cost in some cases. We illustrate this result in the context of hazardous material transportation. However, it could be generalized in similar network design problems as discussed previously.

8. Concluding Remarks

We conclude this paper by suggesting potential future research directions. Providing path-based formulations for robust network design with generalized bounded rationality is an interesting and important future research topic, especially for the cases when a small number of paths are available in the network. While this paper considers road networks without congestion in multi-commodity transportation networks, there is a stream of literature studying boundedly rational

Table 2 Comparison of different network design models

OD	Runs	HNDP-M-Sat		HNDP-SM-Sat		M-Sat/D		SM-Sat/D		M-Sat/SM-Sat	
		# closed	Risk	#closed	Risk	Risk	Diff%	Risk	Diff%	Risk	Diff%
5	1	17	253.0	13	253.0	348.9	37.91	312.7	23.60	308.1	21.78
	2	16	292.3	14	292.3	500.1	71.09	348.2	19.12	302.2	3.39
	3	15	281.1	13	281.1	386.8	37.60	308.6	9.78	365.2	29.92
	4	10	269.3	7	267.6	409.6	52.10	316.0	18.09	361.2	34.98
	5	7	125.0	6	125.0	137.6	10.08	134.3	7.44	131.5	5.20
10	1	15	434.6	14	434.5	564.7	29.94	434.5	0	564.7	29.97
	2	20	489.5	14	489.5	554.9	13.36	489.5	0	554.9	13.36
	3	18	691.4	17	691.4	738.1	6.75	691.4	0	738.1	6.75
	4	18	434.9	17	433.1	593.5	36.47	461.4	6.53	552.8	27.64
	5	19	763.8	17	759.5	810.9	6.17	759.5	0.00	810.9	6.77
15	1	26	895.9	24	892.3	1066.4	19.03	962.2	7.83	996.4	11.67
	2	26	810.8	20	809.5	1098.8	35.52	816.1	0.82	1062.5	31.25
	3	17	645.7	16	641.2	654.5	1.36	641.2	0	654.5	2.07
	4	27	798.6	24	793.8	869.3	8.85	830.9	4.67	832.2	4.84
	5	22	747.3	17	746.7	901.1	20.58	782.3	4.77	841.2	12.66
20	1	25	999.6	19	997.7	1420.3	42.09	997.7	0	1420.3	42.36
	2	26	1177.8	17	1164.0	1359.0	15.38	1273.6	9.42	1262.9	8.50
	3	23	948.9	19	942.6	1060.8	11.79	1021.6	8.38	1058.0	12.24
	4	23	1109.1	19	1109.1	1258.9	13.51	1173.6	5.82	1160.7	4.65
	5	26	1102.4	23	1091.2	1335.8	21.17	1139.1	4.39	1287.9	18.03
25	1	26	1557.8	19	1539.3	1718.4	10.31	1601.2	4.02	1723.6	11.97
	2	22	1537.8	20	1537.8	1592.2	3.54	1582.5	2.91	1646.7	7.08
	3	22	1426.2	19	1416.1	1608.4	12.78	1484.0	4.79	1534.2	8.34
	4	26	1425.5	16	1375.3	1626.5	14.10	1375.3	0	1640.6	19.29
	5	27	1299.0	22	1292.7	1464.6	12.75	1372.3	6.16	1416.6	9.58

user equilibria that considers congestion effects in urban commuting networks. Considering the generalized bounded rationality in user equilibria is certainly a promising future research direction. One more interesting extension would be the consideration of the generalized bounded rationality framework in a noncooperative or cooperative game with satisificing players either with perfect or imperfect information.

Another direction is for solving the robust network design problem. The cutting plane algorithm presented in this paper restricts certain selective paths from being chosen by adding valid cuts iteratively. While the cutting plane algorithm performs well for small and medium size networks, a more efficient algorithm should be developed for large networks (with thousands of nodes and links).

Acknowledgments

This research was partially supported by the National Science Foundation under grant CMMI-1558359.

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Electronic Companion

This Electronic Companion provides additional details of the numerical experiments and supporting materials.

EC.1. Numerical Tests of the Monte-Carlo method and Insights in SM-Sat

We make a comparison of M-Sat and SM-Sat on the Sioux Falls network (shown in Figure EC.1). Let $M = 80\%$ and $\kappa' = 0.1$, $\kappa = 0.1$ and we obtain the following result in Figure EC.2 with OD pair $(3, 19)$. The Sioux Falls network is used since it is easier to analyze the specific paths in detail and computationally efficient to enumerate paths. For M-Sat, P_{cov} is obtained as 95.3% covering paths 1–9 as shown in Table EC.1. For SM-Sat, paths 6–9 are eliminated from M-Sat, resulting in a probability of 84.1%. For path 6, it is an M-Sat path but not an SM-Sat path since it is dominated by paths 1 and 2. Comparing paths 6 and 1, path 1 has subpath $\{5, 6, 8, 16, 17, 19\}$ with cost 15 while path 6 has subpath $\{5, 9, 10, 15, 19\}$ with cost 17. Comparing paths 2 and 6, path

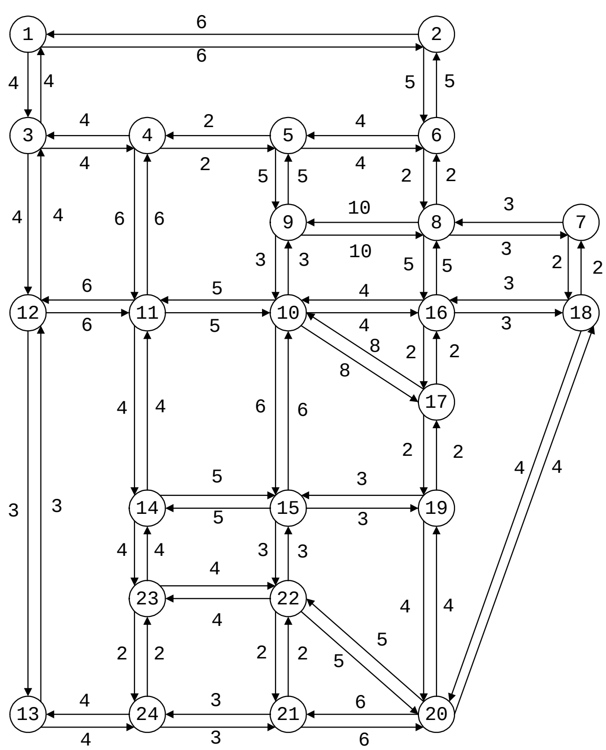


Figure EC.1 Network structure of Sioux Falls

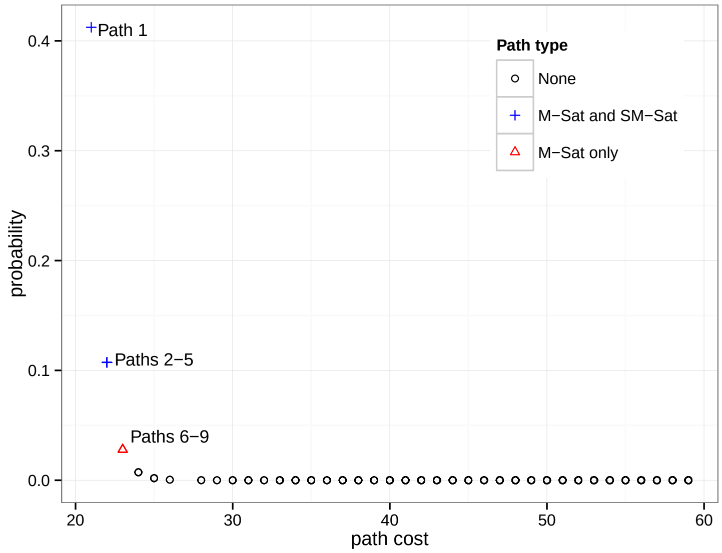


Figure EC.2 Comparison of M-Sat and SM-Sat for OD pair $(3,19)$ of Sioux Falls network

Table EC.1 10 shortest paths from Sioux Falls network of OD pair $(3,19)$

Number	Path Links	Cost	Probability	M-Sat?	SM-Sat?
1	{3, 4, 5, 6, 8, 16, 17, 19}	21	41.24%	✓	✓
2	{3, 4, 5, 9, 10, 16, 17, 19}	22	10.73%	✓	✓
3	{3, 4, 11, 14, 15, 19}	22	10.73%	✓	✓
4	{3, 12, 11, 14, 15, 19}	22	10.73%	✓	✓
5	{3, 12, 13, 24, 21, 22, 15, 19}	22	10.73%	✓	✓
6	{3, 4, 5, 9, 10, 15, 19}	23	2.79%	✓	
7	{3, 4, 11, 10, 16, 17, 19}	23	2.79%	✓	
8	{3, 12, 11, 10, 16, 17, 19}	23	2.79%	✓	
9	{3, 12, 13, 24, 23, 22, 15, 19}	23	2.79%	✓	
10	{3, 4, 5, 6, 8, 7, 18, 16, 17, 19}	24	0.73%		

2 has subpath $\{10,16,17,19\}$ with cost 8 while path 6 has subpath $\{10,15,19\}$ with cost 9. By enforcing a $\kappa = 10\%$ multiplicative indifference band, path 6 violates the definition of SM-Sat. We could achieve similar analysis and results on paths 7–9.

We also randomly generate diverse OD pairs on the Sioux Falls network. We only keep the OD pairs with connected paths and the number of links in the shortest path is no less than 3 for comparison purpose. We record the percentage covered by the M-Sat paths set and the SM-Sat paths set with the probability obtained by RUM considering all paths can be chosen. The results are shown in Table EC.2. The "#links" and "c^0" columns record the number of link and cost of the shortest path without perception error separately. The "Diff %" column records the percentage difference between M-Sat and SM-Sat. The differences are all non-negative since SM-Sat is a subset of M-Sat with the same κ value. We see that with the increase of the bound κ, the covered percentage gets higher. However, the increased probability for M-Sat and SM-Sat could be different, which leads to more diverse covered percentages between M-Sat and SM-Sat. We could also find that with some small κ value (e.g. the values we listed), we are able to cover mostly over 90%. Furthermore, for many OD pairs and κ values, the SM-Sat and M-Sat set is the same for the Sioux Falls network and the average percentage difference is very small. Thus the faster method described for M-Sat still provides a good bound for the SM-Sat.

EC.2. Generating Valid Cuts

Suppose for a certain shipment s, the route solutions of the master problem and the worst case problem are different. Figure EC.3 illustrates this case. The path containing subpath p is the path obtained by the master problem while the path containing subpath $\widehat{p}$ is the path obtained by the worst case problem. p and $\widehat{p}$ are the subpaths that are different. Then we can generate cuts as follows:

$$\sum_{(i,j)\in p} x_{ijs} \le |p| - 1 + z_p, \tag{EC.1}$$

$$z_p \le x_{ijs} \quad \forall (i,j) \in p, \tag{EC.2}$$

$$\sum_{(i,j)\in \widehat{p}} y_{ij} \le |\widehat{p}| - z_p, \tag{EC.3}$$

$$z_p \in \{0,1\} \tag{EC.4}$$

where $|\cdot|$ denotes the number of arcs in the path. The binary variable $z_p = 1$ if the path generated by the master problem contains subpath p. Constraints (EC.1) force z_p to take value 1 if $\sum_{(i,j)\in p} x_{ijs} = |p|$. Constraints (EC.2) ensure $z_p = 0$ if any of its arcs are not used. Constraints (EC.3) make sure to close at least one of the arcs of subpath $\widehat{p}$ if the path generated by the master problem contains subpath p.

Table EC.2 Comparing SM-Sat and M-Sat

O	D	#links	c^0	$\kappa=5\%$		$\kappa=10\%$		$\kappa=15\%$		$\kappa=20\%$	
				M-BR%	Diff%	M-BR%	Diff%	M-BR%	Diff%	M-BR%	Diff%
19	18	3	7	98.27	0	98.27	0	100	0	100	0
19	10	3	8	97.09	0	97.09	0	99.92	0	99.92	0
20	8	3	9	100	0	100	0	100	0	100	0
12	5	3	10	100	0	100	0	100	0	100	0
6	3	3	10	100	0	100	0	100	0	100	0
1	13	3	11	100	0	100	0	100	0	100	0
17	11	3	11	99.37	0	99.37	0	99.37	0	99.96	0.58
11	19	3	12	90.53	0	99.11	0	99.11	0	99.92	0.81
14	5	3	12	100	0	100	0	100	0	100	0
5	14	3	12	100	0	100	0	100	0	100	0
9	14	3	12	99.11	0	99.11	0	99.11	0	100	0.89
23	10	3	13	99.29	0	99.29	0	99.29	0	99.93	0.64
12	9	3	14	88.28	0	100	0	100	0	100	0
19	24	4	11	92.31	0	99.37	7.06	99.37	7.06	99.91	0.54
19	24	4	11	92.31	0	99.37	7.06	99.37	7.06	99.91	0.54
5	18	4	11	99.95	0	99.95	0	99.95	0	99.95	0
19	6	4	11	99.95	0	99.95	0	99.95	0	99.95	0
5	18	4	11	99.95	0	99.95	0	99.95	0	99.95	0
13	15	4	12	91.27	0	99.92	8.65	99.92	8.65	99.92	0
8	3	4	12	100	0	100	0	100	0	100	0
5	17	4	13	89.56	0	99.74	0	99.74	0	99.74	0
23	3	4	13	100	0	100	0	100	0	100	0
21	9	4	14	100	0	100	0	100	0	100	0
22	8	4	14	87.85	0	99.50	0	99.50	0	99.50	0
3	21	4	14	100	0	100	0	100	0	100	0
7	11	4	14	99.53	0	99.53	0	99.53	0	99.53	0
11	7	4	14	99.53	0	99.53	0	99.53	0	99.53	0
3	10	4	14	79.03	0	100	0	100	0	100	0
7	1	4	16	99.50	0	99.50	0	99.50	0	100	0.50
7	1	4	16	99.50	0	99.50	0	99.50	0	100	0.50
13	9	4	17	83.92	0	99.83	0	99.83	0	99.83	0
22	5	4	17	97.78	0	97.78	0	97.78	0	99.78	0.67
16	13	4	18	75.69	0	91.43	0	97.97	3.27	99.33	0.68
3	15	4	19	84.13	0	96.79	6.33	99.65	9.19	99.97	3.18
18	4	5	13	99.84	0	99.84	0	99.84	0	99.84	0
19	5	5	15	84.57	0	97.41	0	99.36	0	99.36	0
7	3	5	15	100	0	100	0	100	0	100	0
5	19	5	15	84.57	0	97.41	0	99.36	0	99.36	0
3	22	5	16	85.40	0	100	14.59	100	14.59	100	0
19	2	5	16	99.42	0	99.42	0	99.42	0	99.91	0.50
8	23	5	18	76.95	0	92.94	0	96.27	3.32	99.03	6.09
23	6	5	20	83.44	0	93.59	0	96.06	2.47	99.06	5.47
14	2	5	21	94.35	0	94.35	0	99.34	4.98	99.77	2.09
24	16	6	15	73.24	0	95.48	11.12	98.86	14.50	98.86	1.69
8	13	6	19	97.90	0	97.90	0	97.90	0	99.02	0
23	2	6	23	81.77	0	95.76	0	97.81	0	98.41	0.60
3	19	7	21	84.15	21.46	95.31	11.16	98.94	9.21	99.89	4.57

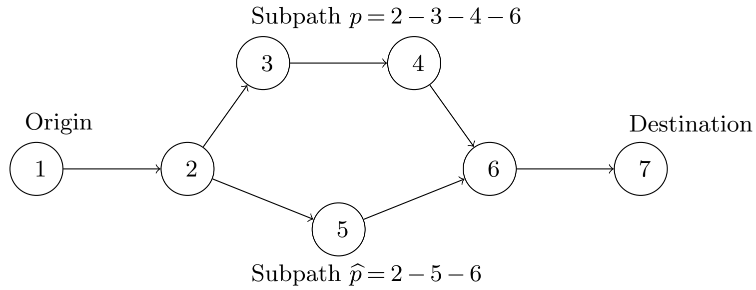
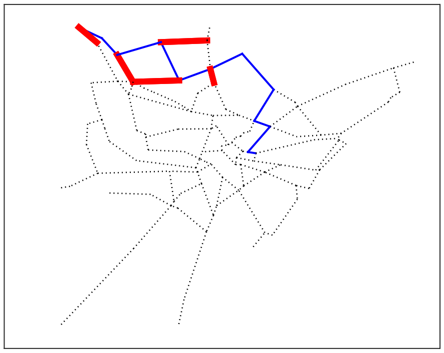


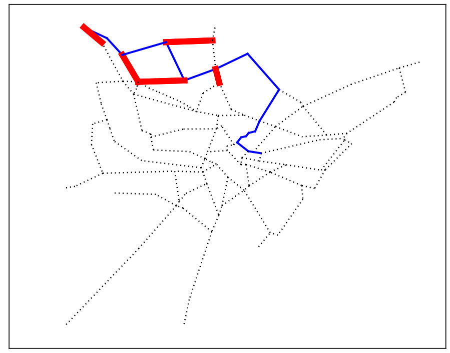
Figure EC.3 A network case when paths are different

EC.3. Algorithmic Process

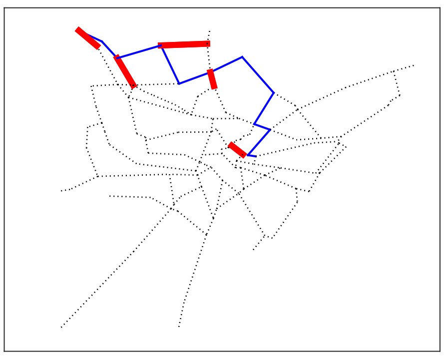
We show the algorithmic process using an illustrative example with 1 OD pair (106, 71) solving the HNDP considering SM-Sat (HNDP-SM-Sat). The WCP is solved by a search algorithm to choose the path having the highest risk value within the SM-Sat path set. The red thicker solid links denote the road segments that are closed. The blue solid links denote the chosen route. For this case, the algorithm takes 27 iterations. Figure EC.4 shows step 24 and the last iteration of the algorithm. R_m and C_m denote the risk and cost of the MP while R_w and C_w denote the risk and cost of the WCP. In iteration 24, five road segments are closed, resulting a minimum risk route with risk value of 767.0. However assuming carriers are subpath multiplicative satisficing, it is possible a worst case route with risk value 1240.8 is chosen. Since the master problem route and worst case route are different, the algorithm compares these two routes and generates cuts. Then it proceeds to the next iteration until the route chosen by the master problem is the same with the worst case problem, as shown in iteration 27 in Figure EC.4c and EC.4d. For all 27 iterations, we find that 16 of them close 5 arcs. The effort of these 15 iterations is to find the critical arcs that could block all the SM-Sat paths. We also record the risk values of MP and WCP for all 27 iterations in Figure EC.5a. We see the MP provides a lower bound while WCP provides an upper bound on the risk. For this particular case, the MP risk value remains the same since there is much flexibility in closing arcs to achieve minimum risk. Based on the closed arcs by MP, WCP obtains a worst case risk. The algorithm stops when MP and WCP achieve the same risk value. More generally, the lower bound risk value becomes larger with more iteration, providing a better lower bound. We provide an example solving a case with 20 OD pairs in Figure EC.5b.



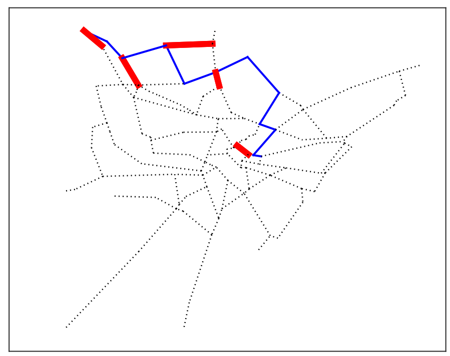
(a) MP network in iteration 24, $R_m = 767.0$, $C_m = 28898$



(b) WCP network in iteration 24, $R_w = 1240.8$, $C_w = 29099$



(c) MP network in iteration 27, $R_m = 767.0$, $C_m = 28898$

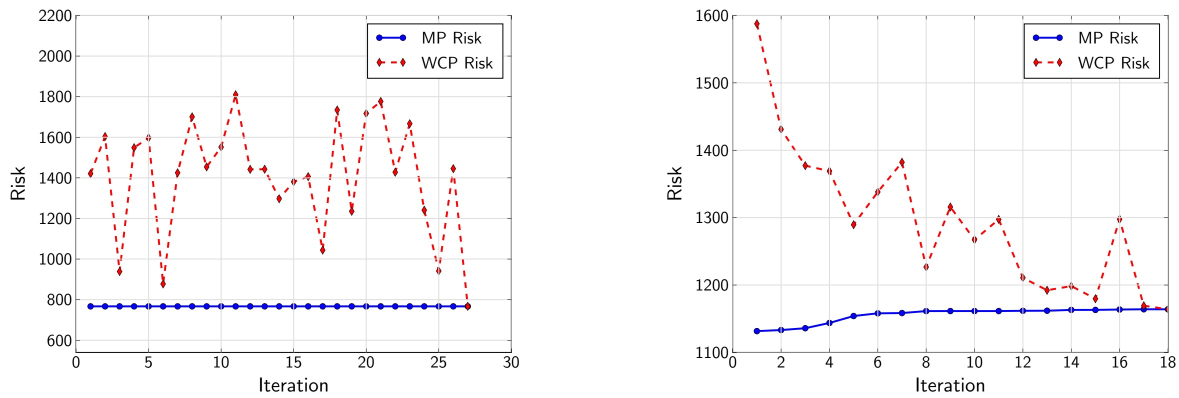


(d) WCP network in iteration 27, $R_w = 767.0$, $C_w = 28898$

Figure EC.4 MP and WCP network at iterations 24 and 27 of the cutting plane algorithm for HNDP-SM-Sat

EC.4. Efficiency of the Algorithm

When we solve both MP and WCP optimally and use the cutting plane algorithm, we are able to obtain an optimal solution. However, we could still solve the robust network design problem heuristically by simple modifications for cases taking too much time. In order to do that, we could set a limit on the MP and WCP solving times and the total algorithm time. WCP provides a solution chosen by the carriers and thus gives a feasible solution to the robust network design problem. Then the best solution among all the ones generated by the WCPs will provide a lowest upper bound. MP gives a solution that is most desired by the upper level and the solution of the



(a) Example 1 with one OD pair

(b) Example 2 with 20 OD pairs

Figure EC.5 Risk values of MP and WCP of all iterations for HNDP-SM-Sat

robust network design problem is no better than this solution. Thus MP provides a lower bound on the solution. By obtaining the highest value among all the MP solutions, we could achieve a best lower bound. Then by comparing the best lower and upper bounds we have, we could obtain a solution with a quantified optimality gap.

We record the time of solving various OD pairs for both the HNDP considering M-Sat (HNDP-M-Sat) and HNDP considering SM-Sat (HNDP-SM-Sat) cases in Table EC.3. We test 5 runs for diverse number of OD pairs: 5, 10, 15, 20 and 25. The test cases are randomly generated such that the number of links between each OD pair is no smaller than 10 and the distance is higher than 15,000. The demand between each OD pair is uniformly generated from the interval $[10, 100]$ and is rounded to the nearest integer. In obtaining the results, we set $\alpha = 0.025$ and $\kappa = 0.05$. The columns "MP Time", "WCP Time" and "Total Time" record the solution time in seconds. The time limit for solving the master problem is set as 1 hour and the total time limit is set as 5 hours. Column "# Iter" shows the number of iterations used for solving each instance.

By observing the "MP Time", "WCP Time" and "Total Time" columns in Table EC.3, we can see that MP accounts for most of the algorithmic effort. WCP is solved by the search algorithm described in Section 6.2 and is easy to solve. For HNDP-SM-Sat, all instances are solved to optimality. For HNDP-M-Sat, one test run (run 4 for OD pair number 25) obtains a sub-optimal solution with

Table EC.3 Time of the algorithm

OD	Runs	HNDP-M-Sat				HNDP-SM-Sat			
		MP Time	WCP Time	Total Time	# Iter	MP Time	WCP Time	Total Time	# Iter
5	1	58.2	24.6	83.3	37	41.0	24.3	66.1	31
	2	62.2	29.2	92.0	35	83.8	27.7	113.2	30
	3	37.3	13.2	50.8	21	48.4	22.1	72.0	26
	4	10.3	4.2	14.5	7	8.0	4.4	13.6	6
	5	33.8	15.3	49.7	22	17.2	10.0	28.4	12
10	1	84.2	32.9	117.5	18	57.3	30.7	89.7	13
	2	175.3	84.1	240.7	40	97.5	51.3	151.7	28
	3	42.6	22.4	65.3	13	41.0	26.3	69.0	13
	4	182.3	64.3	248	40	137.1	65.4	205.2	33
	5	56.5	25.7	82.8	17	45.7	25.4	73.6	15
15	1	153.9	49.2	204.3	20	162.8	59.7	225.3	20
	2	160.7	51.5	213.4	20	153.6	59.8	216.3	20
	3	222.9	63.5	287.8	25	102.9	47.9	153.8	15
	4	179.2	47.7	228.1	18	183.9	63.1	250.2	20
	5	540.9	78.2	621.3	31	762.2	142.4	909.3	39
20	1	227.0	69.4	297.7	17	176.8	81.0	261.3	14
	2	401.5	84.4	487.8	22	275.9	104.9	385.2	18
	3	4999.9	162.7	5267.6	44	1864	165.5	2036.1	32
	4	514.6	108.0	628.2	27	383.9	111.4	500.8	21
	5	2088.1	139.3	2231.9	38	694.3	144.9	845.6	29
25	1	17933.9	136.3	18075.3	31	1369.9	113.4	1488.5	19
	2	1119.5	115.7	1239.0	25	1406.0	165.6	1579.1	30
	3	3760.4	127.6	3893.9	30	2271.5	156.7	2434.6	24
	4	18243.2	141.1	18389.4*	28	4690.1	136.6	4832.7	22
	5	4379.5	183.5	4570.9	39	1567.1	163.3	1738.6	25

* This instance is not solved optimally within time limit and the gap is 2.66%.

a gap between lower and upper bounds of 2.66%. Looking at the column "Total Time", we find that for 17 out of the 25 runs, HNDP-M-Sat takes more time to solve, especially when the instances are harder to solve (OD number of 20 and 25). In some case, obtaining an optimal solution could be too excessive for HNDP-M-Sat. On average, HNDP-M-Sat takes 1593.8 seconds more to solve compared to HNDP-SM-Sat. Additionally, by looking at column "# Iter", we find the M-Sat case tends to have more iterations than the SM-Sat. For 20 out of the 25 runs, HNDP-M-Sat takes more iterations. This is since M-Sat only bounds the sum of the perception error instead of the link specific ones. Thus M-Sat has more flexibility in assigning the perception error and generally results in more iterations.