On the Price of Satisficing in Network User Equilibria

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Abstract

When network users are satisficing decision-makers, the resulting traffic pattern attains a satisficing user equilibrium, which may deviate from the (perfectly rational) user equilibrium. In a satisficing user equilibrium traffic pattern, the total system travel time can be worse than in the case of the PRUE. We show how bad the worst-case satisficing user equilibrium traffic pattern can be, compared to the perfectly rational user equilibrium. We call the ratio between the total system travel times of the two traffic patterns the price of satisficing, for which we provide an analytical bound. Using the sensitivity analysis for variational inequalities, we propose a numerical method to quantify the price of satisficing for any given network instance.

Keywords: bounded rationality; satisficing; user equilibrium; sensitivity analysis

1 Introduction

Instead of assuming a perfectly rational person with a clear system of preferences and perfect knowledge of the surrounding decision-making environment, we can consider *boundedly* rational persons with (1) an ambiguous system of preferences and (2) lack of complete information, following Simon (1955). When decision makers are indifferent among alternatives within a certain threshold, they are called *satisficing* decision makers, opposed to *optimizing* decision makers. The notion of satisficing was first introduced by Simon (1955, 1956). Satisficing decision makers choose any alternative whose utility level is above a threshold, called an *aspiration level*, even when the alternative is not optimal. The satisficing behavior is related to the first source of boundedness—an ambiguous system of preferences.

In transportation research, modeling drivers' route choice is an important task. While the travel-time minimization has been traditionally used as a basis for such modeling, sub-optimal route-choice behavior has gained attention. Since Mahmassani and Chang (1987), bounded rationality has gained attention in the transportation research literature (Szeto and Lo, 2006; Wu et al., 2013;

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Han et al., 2015; Szeto and Lo, 2006; Ge and Zhou, 2012; Di et al., 2014; Guo, 2013; Lou et al., 2010). Empirical evidence supports bounded rationality of drivers (Nakayama et al., 2001; Zhu and Levinson, 2010). The notion of bounded rationality has also been considered in the evaluation of value of times in connection to route-choice modeling (Xu et al., 2017), and in the model of behavior adjustment process (Ye and Yang, 2017). We refer readers to a review of Di and Liu (2016). In the non-transportation literature, the notion of bounded rationality and satisficing has also received much attention (Charnes and Cooper, 1963; Lam et al., 2013; Jaillet et al., 2016; Chen et al., 1997; Brown and Sim, 2009).

While the above-mentioned transportation research literature considers boundedly rational drivers, their discussion is limited to satisficing drivers without considering the second source of boundedness: lack of complete information on the decision environment. Sun et al. (2018) connect the first and the second sources of boundedness by considering both satisficing behavior and incomplete information, in the context of shortest-path finding in *congestion-free* networks. Sun et al. (2018) study the second source by considering errors in drivers' perception of arc travel time, and conclude that their perception-error model can generally capture both sources of boundedness in rationality in a single unified modeling framework.

In the literature, the traditional network user equilibrium, Wardrop equilibrium in particular, is called the perfectly rational user equilibrium (PRUE), while a traffic pattern equilibrated among satisficing drivers is called a boundedly rational user equilibrium (BRUE). In this paper, we will use a new term *satisficing user equilibrium (SatUE)* instead of BRUE to emphasize that it only considers the first source of boundedness without considering drivers' incomplete information on the decision environment. Note that SatUE differs from the stochastic user equilibrium (SUE) (Sheffi, 1985) in two important aspects. First, drivers are assumed to be optimizing decision makers in SUE, while they are satisficing in SatUE. Second, with appropriate probability distributions assumed in the random utility model in SUE, each path possesses a probability of being chosen; hence we can compute the expected traffic flow rate in each path. In SatUE, however, each satisficing path is acceptable to drivers, but it may or may not be chosen by drivers and we do not know its probability of being chosen. See further discussion in Di and Liu (2016).

The main contribution of this paper is the quantification of how bad the total system travel time in SatUE can be. In a SatUE traffic pattern, the total system travel time can be either greater than or less than that of PRUE. We define the *price of satisficing* (PoSat) as the ratio between the worst-case total system travel time of SatUE and the total system travel time of PRUE. This paper quantifies PoSat both *analytically* and *numerically.*

1.1 Analytical Quantification of the Price of Satisficing

The analytical quantification of PoSat is related to the price of anarchy (PoA) (Koutsoupias and Papadimitriou, 1999; Roughgarden and Tardos, 2002) that compares the performances of the system optimal solutions and the PRUE solutions. Using a similar idea, we can also compare the performance of the perfectly rational user equilibrium traffic patterns and satisficing user equilibrium traffic patterns. While PoA quantifies how much system-wide performance we can lose by competing, PoSat quantifies how much we can lose by satisficing. Roughgarden and Tardos (2002) define and study the PoA of *approximate Nash equilibria*, which are essentially SatUE patterns. We develop the bound on PoSat by learning from the PoA of approximate Nash equilibria (Christodoulou et al., 2011) and incorporating the ideas from the sensitivity analysis of traffic equilibria (Dafermos and Nagurney, 1984) with a novel technique. Note that Perakis (2007) studies the PoA of the *exact* Nash equilibria with general nonlinear, asymmetric cost functions.

The notion of PoSat is also related to the *price of risk aversion* (Nikolova and Stier-Moses, 2015) and the *deviation ratio* (Kleer and Schäfer, 2016). When network users are risk-averse decision makers, the price of risk aversion compares the performances of the resulting equilibrium among risk-averse users and the (risk-neutral) PRUE. When network users' cost functions are deviated from the true cost functions for some reasons, the deviation ratio compares the performances of the resulting equilibrium and the PRUE. Kleer and Schäfer (2016) show that the burden of risk aversion is a special case of the deviation ratio. In both research articles, however, only cases with a common single origin node are considered. In this paper, we consider general cases with multiple origin nodes and multiple destination nodes, with asymmetric travel time functions.

1.2 Numerical Quantification of the Price of Satisficing

The numerical quantification of PoSat utilizes the sensitivity analysis of parametric variational inequalities. We can present sufficient conditions for SatUE in the form of parametric variational inequalities, which we call the user equilibrium with perception errors (UE-PE). In particular, we consider variational inequality problems of the following form: to find $\bar{\boldsymbol{v}} \in \boldsymbol{V}$ such that

$$\widetilde{\boldsymbol{t}}(\bar{\boldsymbol{v}}, \boldsymbol{\lambda})^\top (\boldsymbol{v} - \bar{\boldsymbol{v}}) \geq 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V} \tag{1}$$

for some $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, where $\boldsymbol{V} \subset \mathbb{R}^n$, $\boldsymbol{\Lambda} \subset \mathbb{R}^n$, and $\widetilde{\boldsymbol{t}} : \boldsymbol{V} \times \boldsymbol{\Lambda} \mapsto \mathbb{R}^n$. Note that $\boldsymbol{\lambda}$ denotes the perception parameter. When a network instance is given, we can compute the worst-case performance of SatUE, by varying the perception parameters in the UE-PE model. This method requires information on derivatives of equilibrium solutions with respect to the perception parameters. When the Jacobian of the travel time function is symmetric, the corresponding variational inequality problem admits an equivalent convex optimization reformulation; hence, one can adopt sensitivity analysis results on convex optimization for computing PoSat. On the other hand, when the Jacobian is asymmetric, such an equivalent convex optimization problem cannot be formulated; rather we need to work with the variational inequality form.

Tobin and Friesz (1988) first compute derivatives in the sensitivity analysis of traffic equilibria, using the implicit function theorem and applying ideas from sensitivity analysis for nonlinear programming problems. Friesz et al. (1990) apply the derivative result in solving mathematical programming with equilibrium constraints (MPEC). Later, using an affine variable transformation, Patriksson and Rockafellar (2003) and Patriksson (2004) provide derivatives under weaker regularity conditions for traffic equilibria, and Josefsson and Patriksson (2007) apply the method to network design problems. Chung et al. (2014) compare methods of sensitivity analysis and discuss the conditions under which the method of Tobin and Friesz (1988) is valid. We emphasize that we can apply most sensitivity analysis results, since our formulation of UE-PE is in a parametric variational inequality form. In this paper, we develop our method based on the sensitivity-based method of Patriksson (2004).

1.3 Organization of the Paper

This paper is organized as follows. In Section 2, we introduce the notation and define various concepts including user equilibrium, system optimum, satisficing behavior, price of anarchy, and price of satisficing. In Section 3, we define the user equilibrium with perception errors and make connections with satisficing user equilibrium. Our main result is introduced in Section 4, where we derive the analytical worst-case bound on the price of satisficing. In Section 5, we show that a gradient projection method based on sensitivity analysis for variational inequalities can be applied to compute the worst-case performances of satisficing user equilibrium, and demonstrate our results through numerical examples. Section 6 concludes this paper.

2 Notation and Definitions

Since we will use path-based and arc-based flow variables and their corresponding functions and sets interchangeably, we need clear definitions of variables, sets, and functions. We use boldfaced lower-case letters for vector quantities as in $\boldsymbol{v}$ and normal lower-case letters for their components as in v_a; similarly, vector-valued functions like $\boldsymbol{t}(\cdot)$ and their components like $t_a(\cdot)$. We use boldfaced upper-case letters for the set that they belong to, as in $\boldsymbol{v} \in \boldsymbol{V}$. We use calligraphic capital letters for sets of indices as in $\mathcal{N}$. The only exception is a vector Q with Q_w being its elements; we use q_i^w for another value related to Q.

2.1 Traffic Flow Variables and Feasible Sets

We consider a network with a set of origin and destination $\mathcal{W}$ that is represented by directed graph $G(\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ is the set of nodes, and $\mathcal{A}$ is the set of arcs. For each OD pair $w \in \mathcal{W}$, the travel demand is Q_w and the set of available paths is $\mathcal{P}_w$. The set of all available paths in the whole network is defined as $\mathcal{P} = \cup_{w \in \mathcal{W}} \mathcal{P}_w$.

We also define the set of path flow variables $\boldsymbol{f}$ as

$$\boldsymbol{F} = \left\{ \boldsymbol{f} : \sum_{p \in \mathcal{P}_w} f_p = Q_w \quad \forall w \in \mathcal{W}, \qquad f_p \geq 0 \quad \forall p \in \mathcal{P} \right\}$$

and the corresponding set of arc flow variables $\boldsymbol{v}$ is defined as

$$\boldsymbol{V} = \left\{ \boldsymbol{v} : v_a = \sum_{p \in \mathcal{P}} \delta_a^p f_p \quad \forall a \in \mathcal{A}, \qquad \boldsymbol{f} \in \boldsymbol{F} \right\}$$

where $\delta_a^p = 1$ if path p contains arc a and $\delta_a^p = 0$ otherwise. Let $\mathcal{A}_i^+$ and $\mathcal{A}_i^-$ be the set of arcs whose tail node and head node are i, respectively. When we need to preserve OD information in arc flow variables, we use $\boldsymbol{x}$ as follows:

$$\begin{aligned} \boldsymbol{X} &= \left\{ \boldsymbol{x} : x_a^w = \sum_{p \in \mathcal{P}_w} \delta_a^p f_p \quad \forall a \in \mathcal{A}, w \in \mathcal{W} \qquad \boldsymbol{f} \in \boldsymbol{F} \right\} \\ &= \left\{ \boldsymbol{x} : \sum_{a \in \mathcal{A}_i^+} x_a^w - \sum_{a \in \mathcal{A}_i^-} x_a^w = q_i^w \quad \forall w \in \mathcal{W}, i \in \mathcal{N} \right\} \end{aligned}$$

where $q_i^w = -Q_w$ if $i = o(w)$, $q_i^w = Q_w$ if $i = d(w)$, and $q_i^w = 0$ otherwise.

We have $v_a = \sum_{p \in \mathcal{P}} \delta_a^p f_p$, $x_a^w = \sum_{p \in \mathcal{P}_w} \delta_a^p f_p$, and $v_a = \sum_{w \in \mathcal{W}} x_a^w$. Therefore, the transformations from $\boldsymbol{f}$ to $\boldsymbol{v}$, from $\boldsymbol{f}$ to $\boldsymbol{x}$, and from $\boldsymbol{x}$ to $\boldsymbol{v}$ are unique, which are denoted by $\boldsymbol{f} \mapsto \boldsymbol{v}$, $\boldsymbol{f} \mapsto \boldsymbol{x}$, and $\boldsymbol{x} \mapsto \boldsymbol{v}$, respectively. The inverse transformations are, however, not unique. In the rest of this paper, to emphasize the non-uniqueness of the transformation and refer to *any* result of such transformation, we use $\overset{\text{any}}{\longmapsto}$; for example, with $\boldsymbol{v} \overset{\text{any}}{\longmapsto} \boldsymbol{f}$, we consider any $\boldsymbol{f}$ such that $v_a = \sum_{p \in \mathcal{P}} \delta_a^p f_p$.

We will use $\boldsymbol{v}$, $\boldsymbol{f}$, and $\boldsymbol{x}$ interchangeably to describe the same traffic pattern. In particular, we define

- $\boldsymbol{f}^*$, $\boldsymbol{v}^*$, $\boldsymbol{x}^*$: system optimal flow vectors (Section 2.2)
- $\boldsymbol{f}^0$, $\boldsymbol{v}^0$, $\boldsymbol{x}^0$: perfectly rational user equilibrium flow vectors (Section 2.3)
- $\boldsymbol{f}^\kappa$, $\boldsymbol{v}^\kappa$, $\boldsymbol{x}^\kappa$: (multiplicative) satisficing user equilibrium flow vectors with κ (Section 2.4)

Note that when $\kappa = 0$, we have $\boldsymbol{f}^\kappa = \boldsymbol{f}^0$.

2.2 Travel Time Functions and System Optimum

We denote arc travel function with arc traffic volume $\boldsymbol{v}$ by $t_a(\boldsymbol{v})$ for each arc $a \in \mathcal{A}$. We consider a performance function for each arc a as

$$z_a(\boldsymbol{v}) = t_a(\boldsymbol{v}) v_a.$$

We denote the travel time function along path p with flow $\boldsymbol{f}$ by $c_p(\boldsymbol{f})$. When problems are stated with respect to $\boldsymbol{x}$, given $v_a = \sum_w x_a^w$, we define $\tau_a^w(\boldsymbol{x}) = \tau_a(\boldsymbol{x}) = t_a(\boldsymbol{v})$. We can consider path-based performance function as follows

$$z_p(\boldsymbol{f}) = c_p(\boldsymbol{f}) f_p.$$

Arc travel function and flow travel function are related to each other:

$$c_p(\boldsymbol{f}) = \sum_{a\in\mathcal{A}} \delta_a^p t_a(\boldsymbol{v}).$$

We define the arc-based total system performance function $Z(\boldsymbol{v})$ and path-based total system performance function $C(\boldsymbol{f})$ interchangeably as follows:

$$\begin{aligned} Z(\boldsymbol{v}) &\equiv \sum_{a\in\mathcal{A}} z_a(\boldsymbol{v}) = \sum_{a\in\mathcal{A}} t_a(\boldsymbol{v}) v_a \\ &= \sum_{p\in\mathcal{P}} z_p(\boldsymbol{f}) = \sum_{p\in\mathcal{P}} c_p(\boldsymbol{f}) f_p = \sum_{w\in\mathcal{W}} \sum_{p\in\mathcal{P}_w} c_p(\boldsymbol{f}) f_p \equiv C(\boldsymbol{f}), \end{aligned}$$

which is also called the total system travel time. If a flow pattern minimizes $Z(\cdot)$ or $C(\cdot)$, it is called a *system optimal* flow pattern.

The vector valued function $\boldsymbol{t}(\cdot)$ is called *monotone* in $\boldsymbol{V}$ if

$$[\boldsymbol{t}(\boldsymbol{v}^1) - \boldsymbol{t}(\boldsymbol{v}^2)]^\top (\boldsymbol{v}^1 - \boldsymbol{v}^2) \geq 0 \tag{2}$$

for all $\boldsymbol{v}^1, \boldsymbol{v}^2 \in \boldsymbol{V}$. If (2) holds as a strict inequality for all $\boldsymbol{v}^1 \neq \boldsymbol{v}^2$, it is said *strictly monotone*. The function $\boldsymbol{t}(\cdot)$ is called *strongly monotone* in $\boldsymbol{V}$ with modulus $\alpha > 0$ if

$$[\boldsymbol{t}(\boldsymbol{v}^1) - \boldsymbol{t}(\boldsymbol{v}^2)]^\top (\boldsymbol{v}^1 - \boldsymbol{v}^2) \geq \alpha \left\| \boldsymbol{v}^1 - \boldsymbol{v}^2 \right\|_{\boldsymbol{V}}^2 \tag{3}$$

for all $\boldsymbol{v}^1, \boldsymbol{v}^2 \in \boldsymbol{V}$, where $\|\cdot\|_{\boldsymbol{V}}$ is the l^2-norm in $\boldsymbol{V}$. The monotonicity of path-based travel time function $c_p(\cdot)$ or its vector form $\boldsymbol{c}(\cdot)$ can be similarly defined. The path-based function $c_p(\cdot)$, however, is not strongly monotone in general (e.g., see Example 3 in de Palma and Nesterov, 1998).

2.3 Perfectly Rational User Equilibrium

When network users are perfectly rational, i.e. they seek the shortest path, we attain the perfectly rational user equilibrium (PRUE) defined as follows:

Definition 1 (Perfectly Rational User Equilibrium)**.** A traffic pattern $\boldsymbol{f}^0$ is called a *perfectly rational user equilibrium* (PRUE), if

$$\textsf{(PRUE)} \qquad f_p^0 > 0 \implies c_p(\boldsymbol{f}^0) = \min_{p'\in\mathcal{P}_w} c_{p'}(\boldsymbol{f}^0) \tag{4}$$

for all $p \in \mathcal{P}_w$ and $w \in \mathcal{W}$.

Using the arc travel function, the above condition can be restated as follows

$$f_p^0 > 0 \implies \sum_{a\in\mathcal{A}} \delta_a^p t_a(\boldsymbol{v}^0) = \min_{p'\in\mathcal{P}_w} \sum_{a\in\mathcal{A}} \delta_a^{p'} t_a(\boldsymbol{v}^0) \tag{5}$$

for all $p \in \mathcal{P}_w$ and $w \in \mathcal{W}$.

It is well known that a solution to the following variational inequality problem is a user equilibrium traffic flow (Smith, 1979; Dafermos, 1980):

$$\text{to find } \bar{\boldsymbol{f}} \in \boldsymbol{F} : \sum_{p \in \mathcal{P}} c_p(\bar{\boldsymbol{f}})(f_p - \bar{f}_p) \geq 0 \qquad \forall \boldsymbol{f} \in \boldsymbol{F}, \tag{6}$$

which can be equivalently rewritten as:

$$\text{to find } \bar{\boldsymbol{v}} \in \boldsymbol{V} : \sum_{a \in \mathcal{A}} t_a(\bar{\boldsymbol{v}})(v_a - \bar{v}_a) \geq 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{7}$$

or

$$\text{to find } \bar{\boldsymbol{x}} \in \boldsymbol{X} : \sum_{a \in \mathcal{A}} \sum_{w \in \mathcal{W}} \tau_a(\bar{\boldsymbol{x}})(x_a^w - \bar{x}_a^w) \geq 0 \qquad \forall \boldsymbol{x} \in \boldsymbol{X} \tag{8}$$

with $\tau_a^w(\boldsymbol{x}) = \tau_a(\boldsymbol{x}) = t_a(\boldsymbol{v})$.

With strictly monotone functions $t_a(\cdot)$, the solution $\bar{\boldsymbol{v}}$ to (7) is unique. While the transformations $\bar{\boldsymbol{v}} \overset{\text{any}}{\longmapsto} \bar{\boldsymbol{f}}$ and $\bar{\boldsymbol{v}} \overset{\text{any}}{\longmapsto} \bar{\boldsymbol{x}}$ are not unique, any such $\bar{\boldsymbol{f}}$ and $\bar{\boldsymbol{x}}$ are solutions to (6) and (8), respectively; therefore, solutions to (6) and (8) are not unique in general.

When the travel time on arc a is a function of only v_a, i.e. $t_a = t_a(v_a)$, then it is called *separable*. With separable arc travel time functions, the variational inequality problem (7) admits an equivalent convex optimization problem as formulated by (Beckmann et al., 1956). In general, if the Jacobian matrix of the arc travel time function vector $\boldsymbol{t}(\boldsymbol{v})$ is symmetric, that is,

$$\frac{\partial t_a(\boldsymbol{v})}{\partial v_e} = \frac{\partial t_e(\boldsymbol{v})}{\partial v_a} \qquad \forall a, e \in \mathcal{A},$$

for all $\boldsymbol{v} \in \boldsymbol{V}$, the variational inequality problem (7) can be reformulated as such an equivalent Beckmann-type convex optimization problem (Patriksson, 2015; Friesz and Bernstein, 2016). When the Jacobian is asymmetric, we cannot reformulate (7) as a Beckmann-type convex optimization problem in general; in which case, we call the arc travel time functions *asymmetric* and we work with the variational inequality form (7) to find and analyze PRUE.

2.4 Satisficing User Equilibrium

We introduce definitions of satisficing behavior and corresponding user equilibrium traffic patterns. A typical definition in the transportation research literature (e.g. Lou et al., 2010; Di et al., 2013; Han et al., 2015), termed boundedly rational user equilibrium (BRUE), uses an additive term. We call it additive satisficing user equilibrium, since bounded rationality could imply a broader concept than just satisficing behavior.

Definition 2 (Additive Satisficing)**.** A traffic pattern $\boldsymbol{f}$ is called an *additive satisficing user* *equilibrium (ASatUE)* with E, if

$$(\mathsf{ASatUE}) \qquad f_p > 0 \implies c_p(\boldsymbol{f}) \le \min_{p' \in \mathcal{P}_w} c_{p'}(\boldsymbol{f}) + E \tag{9}$$

for all $p \in \mathcal{P}_w$ and $w \in \mathcal{W}$, where E is a positive constant.

We can also derive a similar definition using a multiplicative term. While the additive form in Definition 2 is popularly used in the transportation research literature, the multiplicative form in Definition 3 enables us to consider the satisficing level in disaggregate link levels as we will observe in this paper.

Definition 3 (Multiplicative Satisficing)**.** A traffic pattern $\boldsymbol{f}^\kappa$ is called a *multiplicative satisficing user equilibrium* with κ, or κ-MSatUE, if

$$(\mathsf{MSatUE}) \qquad f_p^\kappa > 0 \implies c_p(\boldsymbol{f}^\kappa) \le (1+\kappa) \min_{p' \in \mathcal{P}_w} c_{p'}(\boldsymbol{f}^\kappa) \tag{10}$$

for all $p \in \mathcal{P}_w$ and $w \in \mathcal{W}$, where $\kappa \in [0, 1]$ is a constant.

Note that E in (9) and κ in (10) may be defined for each OD pair w, for example E_w and κ_w respectively, to allow non-homogeneous satisficing threshold for each OD pair w. In such cases, however, we assume that travelers for the same OD pair are homogeneous with the same threshold κ_w. In this paper, for simplicity, we use a single value of κ for all OD pairs.

2.5 Price of Satisficing

The price of anarchy (PoA) compares the performances of approximate Nash equilibrium and system optimum; $C(\boldsymbol{f}^\kappa)$ and $C(\boldsymbol{f}^*)$, respectively. Among possibly multiple approximate Nash equilibrium traffic patterns, we are interested in the worst-case. Let us denote the set of approximate Nash equilibrium for a given network instance ρ by $\Psi_\kappa(\rho)$. We can define PoA of a network instance ρ as follows:

$$\mathsf{PoA}(\rho) = \max_{\boldsymbol{f}^\kappa \in \Psi_\kappa(\rho)} \frac{C(\boldsymbol{f}^\kappa)}{C(\boldsymbol{f}^*)}, \tag{11}$$

and we are usually interested in its upper bound among all network instances, $\sup_\rho \mathsf{PoA}(\rho)$.

In the context of bounded rationality and satisficing, we are more interested in comparing the performances of approximate Nash equilibrium—equivalently MSatUE—and the perfectly rational user equilibrium; $C(\boldsymbol{f}^\kappa)$ and $C(\boldsymbol{f}^0)$, respectively. We define the price of satisifcing (PoSat) of instance ρ as follows:

$$\mathsf{PoSat}(\rho) = \max_{\boldsymbol{f}^\kappa \in \Psi_\kappa(\rho)} \frac{C(\boldsymbol{f}^\kappa)}{C(\boldsymbol{f}^0)}, \tag{12}$$

and its upper bound among all network instances, $\sup_\rho \mathsf{PoSat}(\rho)$.

3 User Equilibrium with Perception Errors

Related to MSatUE, we introduce the user equilibrium with perception error (UE-PE) model. In this model, we assume that network users are optimizing, i.e. seeking the shortest path; however, we assume that users may have their own perception of the travel time function.

We let ε_a^w denote the perception error of travel time along arc a of users in OD pair w. A vector $\bar{\boldsymbol{x}} \in \boldsymbol{X}$ is a solution to the UE-PE model, if

$$\sum_{a\in\mathcal{A}}\sum_{w\in\mathcal{W}}(t_a(\bar{\boldsymbol{v}}) - \varepsilon_a^w)(x_a^w - \bar{x}_a^w) \geq 0 \qquad \forall \boldsymbol{x} \in \boldsymbol{X} \tag{13}$$

for *some* $\boldsymbol{\varepsilon}$ such that

$$0 \leq \varepsilon_a^w \leq \frac{\kappa}{1+\kappa} t_a(\bar{\boldsymbol{v}}) \qquad \forall a \in \mathcal{A}, w \in \mathcal{W}. \tag{14}$$

We note that $t_a(\bar{\boldsymbol{v}}) - \varepsilon_a^w$ is the *perceived* travel time for drivers of OD pair w. The term ε_a^w represents the perception error for arc a and OD pair w. In this model, we assume all drivers for each OD pair are homogeneous in their perception of arc travel time.

With changes of variables $\lambda_a^w t_a(\boldsymbol{v}) = t_a(\boldsymbol{v}) - \varepsilon_a^w$, the UE-PE model (13) can be restated as follows:

$$(\mathsf{UE\text{-}PE\text{-}}\boldsymbol{X}) \qquad \sum_{a\in\mathcal{A}}\sum_{w\in\mathcal{W}}\lambda_a^w t_a(\bar{\boldsymbol{v}})(x_a^w - \bar{x}_a^w) \geq 0 \qquad \forall \boldsymbol{x} \in \boldsymbol{X} \tag{15}$$

for some $\lambda_a^w \in [\frac{1}{1+\kappa}, 1]$ for all $w \in \mathcal{W}$ and $a \in \mathcal{A}$. We observe that the UE-PE model generates a subset of SMSatUE traffic flow patterns.

Lemma 1 ($\mathsf{UE\text{-}PE\text{-}}\boldsymbol{X} \implies$ MSatUE)**.** *Suppose $\bar{\boldsymbol{x}}$ is a solution to* $\mathsf{UE\text{-}PE\text{-}}\boldsymbol{X}$ *in* (13) *with some $\bar{\boldsymbol{\lambda}}$ where $\bar{\lambda}_a^w \in [\frac{1}{1+\kappa}, 1]$ for all $w \in \mathcal{W}$ and $a \in \mathcal{A}$. Then any $\bar{\boldsymbol{f}}$ with $\bar{\boldsymbol{x}} \xmapsto{\text{any}} \bar{\boldsymbol{f}}$ is a κ-MSatUE flow.*

Proof of Lemma 1. Given $\bar{\boldsymbol{f}}$, we let $\bar{\boldsymbol{v}}$ be the arc flow vector from $\bar{\boldsymbol{f}} \mapsto \bar{\boldsymbol{v}}$. Let $\bar{\varepsilon}$ is the perception error that makes $\bar{\boldsymbol{x}}$ a solution to (13). Under the perception error $\bar{\varepsilon}$, we know that $\bar{\boldsymbol{x}}$ is a user equilibrium traffic flow; hence the following condition holds from (5):

$$\bar{f}_p > 0 \implies \sum_{a\in\mathcal{A}} \delta_a^p \bar{\lambda}_a^w t_a(\bar{\boldsymbol{v}}) = \min_{p'\in\mathcal{P}_w} \sum_{a\in\mathcal{A}} \delta_a^{p'} \bar{\lambda}_a^w t_a(\bar{\boldsymbol{v}}) \tag{16}$$

for all $p \in \mathcal{P}_w$ and $w \in \mathcal{W}$. Since $\bar{\lambda}_a^w \in [\frac{1}{1+\kappa}, 1]$, the right-hand-side of (16) implies

$$\frac{1}{1+\kappa}\sum_{a\in\mathcal{A}} \delta_a^p t_a(\bar{\boldsymbol{v}}) \leq \min_{p'\in\mathcal{P}_w} \sum_{a\in\mathcal{A}} \delta_a^{p'} \bar{\lambda}_a^w t_a(\bar{\boldsymbol{v}}) \leq \min_{p'\in\mathcal{P}_w} \sum_{a\in\mathcal{A}} \delta_a^{p'} t_a(\bar{\boldsymbol{v}}),$$

which is equivalent to the following path flow form:

$$c_p(\bar{\boldsymbol{f}}) \leq (1+\kappa) \min_{p'\in\mathcal{P}_w} c_{p'}(\bar{\boldsymbol{f}}).$$

Therefore, we conclude that $\bar{\boldsymbol{f}}$ is a κ-MSatUE traffic flow. $\square$

We can also provide a path-based formulation of UE-PE:

$$(\textsf{UE-PE-}\boldsymbol{F}) \qquad \sum_{w\in\mathcal{W}}\sum_{p\in\mathcal{P}_w} \widetilde{c}_p^w(\bar{\boldsymbol{f}})(f_p - \bar{f}_p) \geq 0 \qquad \forall \boldsymbol{f} \in \boldsymbol{F} \tag{17}$$

for the perceived path travel time functions $\widetilde{c}_p^w(\boldsymbol{f}) = \sum_{a\in\mathcal{A}} \delta_a^p \lambda_a^w t_a(\boldsymbol{v})$ with *some* $\lambda_a^w \in [\frac{1}{1+\kappa}, 1]$.

Lemma 2 (UE-PE-$\boldsymbol{F}$ $\iff$ UE-PE-$\boldsymbol{X}$)**.** *If $\bar{\boldsymbol{f}} \in \boldsymbol{F}$ is a solution to* UE-PE-$\boldsymbol{F}$ *in* (17)*, then $\bar{\boldsymbol{x}}$ with $\bar{\boldsymbol{f}} \mapsto \bar{\boldsymbol{x}}$ is a solution to* UE-PE-$\boldsymbol{X}$ *in* (15)*. Conversely, if $\bar{\boldsymbol{x}} \in \boldsymbol{X}$ is a solution to* UE-PE-$\boldsymbol{X}$ *in* (15)*, then any $\bar{\boldsymbol{f}}$ with $\bar{\boldsymbol{x}} \xmapsto{\text{any}} \bar{\boldsymbol{f}}$ is a solution to* UE-PE-$\boldsymbol{F}$ *in* (17)*.*

Proof of Lemma 2. We can prove both directions by observing that

$$\begin{aligned}
\sum_{w\in\mathcal{W}}\sum_{p\mathcal{P}_w} \widetilde{c}_p^w(\bar{\boldsymbol{f}})(f_p - \bar{f}_p) &= \sum_{w\in\mathcal{W}}\sum_{p\in\mathcal{P}_w}\sum_{a\in\mathcal{A}} \delta_a^p \lambda_a^w t_a(\bar{\boldsymbol{v}})(f_p - \bar{f}_p) \\
&= \sum_{w\in\mathcal{W}}\sum_{a\in\mathcal{A}} \lambda_a^w t_a(\bar{\boldsymbol{v}}) \bigg(\sum_{p\in\mathcal{P}_w} \delta_a^p f_p - \sum_{p\in\mathcal{P}_w} \delta_a^p \bar{f}_p \bigg) \\
&= \sum_{w\in\mathcal{W}}\sum_{a\in\mathcal{A}} \lambda_a^w t_a(\bar{\boldsymbol{v}})(x_a^w - \bar{x}_a^w).
\end{aligned}$$

□

When the values of λ_a^w are the same across all $w \in \mathcal{W}$, i.e. $\lambda_a = \lambda_a^w$ for all $w \in \mathcal{W}$, we can simplify (15) as follows:

$$(\textsf{UE-PE-}\boldsymbol{V}) \qquad \sum_{a\in\mathcal{A}} \lambda_a t_a(\bar{\boldsymbol{v}})(v_a - \bar{v}_a) \geq 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V} \tag{18}$$

for *some* $\lambda_a \in [\frac{1}{1+\kappa}, 1]$ for each $a \in \mathcal{A}$. The simplified model (18) has been considered in the literature for approximate Nash equilibrium (Christodoulou et al., 2011) and Nash equilibrium with deviated travel time functions (Kleer and Schäfer, 2016). For the simplified model, we can state:

Lemma 3 (UE-PE-$\boldsymbol{V}$ $\implies$ UE-PE-$\boldsymbol{X}$)**.** *Suppose that $\bar{v} \in \boldsymbol{V}$ is a solution to* UE-PE-$\boldsymbol{V}$ *in* (18)*. Let $\bar{\boldsymbol{x}}$ be any vector with $\bar{\boldsymbol{v}} \xmapsto{\text{any}} \bar{\boldsymbol{x}}$. Then $\bar{\boldsymbol{x}}$ is a solution to* UE-PE-$\boldsymbol{X}$ *in* (15)*.*

While Lemmas 1, 2, and 3 provide sufficient conditions for a traffic flow pattern to be a κ-MSatUE, Theorem 1 of Christodoulou et al. (2011) provides a necessary condition. Although Christodoulou et al. (2011) assumed separable arc travel time functions, their proof is still valid for nonseparable travel time functions.

Lemma 4 (A necessary condition of MSatUE)**.** *Let $\boldsymbol{f}^\kappa \in \boldsymbol{F}$ be a κ-MSatUE and $\boldsymbol{v}^\kappa \in \boldsymbol{V}$ be the corresponding arc flow vector with $\boldsymbol{f}^\kappa \mapsto \boldsymbol{v}^\kappa$. Then we have*

$$\sum_{a\in\mathcal{A}} t_a(\boldsymbol{v}^\kappa)((1+\kappa)v_a - v_a^\kappa) \geq 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{19}$$

Christodoulou et al. (2011) derive a tight bound on the price of anarchy on approximate Nash equilibria based on Lemma 4. We conclude this section:

Theorem 1. *Implications in Lemmas 1–4 are summarized as follows:*

$$
\begin{array}{ccccccc}
\mathsf{UE\text{-}PE\text{-}}\boldsymbol{V} & \Longrightarrow & \mathsf{UE\text{-}PE\text{-}}\boldsymbol{X} & \Longrightarrow & \text{MSatUE} & \Longrightarrow & (19), \\
 & & \Updownarrow & & & & \\
 & & \mathsf{UE\text{-}PE\text{-}}\boldsymbol{F} & & & &
\end{array}
$$

where $\mathsf{X} \implies \mathsf{Y}$ *means that any solution to* X *yields a solution to* Y.

4 Bounding the Price of Satisficing

We first provide analytical bounds of $C(\boldsymbol{f}^\kappa)$ compared to $C(\boldsymbol{f}^0)$.

4.1 Lessons from the Price of Anarchy

We first observe that $\mathsf{PoSat}(\rho) \leq \mathsf{PoA}(\rho)$ for all network instance ρ, since $C(\boldsymbol{f}^0) \geq C(\boldsymbol{f}^*)$. This enables us to use the results from the price of anarchy literature for bounding PoSat. Theorem 2 of Christodoulou et al. (2011) provides the price of anarchy for the general polynomial cases, which immediately leads to the following result:

Lemma 5. *Suppose $\boldsymbol{f}^\kappa$ is a κ-MSatUE flow, and $t_a(\cdot)$ is a polynomial with degree n. Define*

$$
\zeta(\kappa, n) = \begin{cases} (1+\kappa)^{(n+1)} & \text{if } \kappa \geq (n+1)^{1/n} - 1, \\ \left(\frac{1}{1+\kappa} - \frac{n}{(n+1)^{(n+1)/n}} \right)^{-1} & \text{if } 0 \leq \kappa \leq (n+1)^{1/n} - 1. \end{cases} \tag{20}
$$

Then we have

$$
C(\boldsymbol{f}^*) \leq C(\boldsymbol{f}^\kappa) \leq \zeta(\kappa, n) C(\boldsymbol{f}^*) \leq \zeta(\kappa, n) C(\boldsymbol{f}^0). \tag{21}
$$

That is, the PoSat is bounded above by $\zeta(\kappa, n)$.

Proof of Lemma 5. From Theorem 2 of Christodoulou et al. (2011), we have

$$
C(\boldsymbol{f}^\kappa) \leq \zeta(\kappa, n) C(\boldsymbol{f}) \quad \forall \boldsymbol{f} \in \boldsymbol{F}. \tag{22}
$$

Picking $\boldsymbol{f} = \boldsymbol{f}^0$ in (22), we obtain the upper bound on $C(\boldsymbol{f}^\kappa)$. Inequalities involving $C(\boldsymbol{f}^*)$ are from the fact $C(\boldsymbol{f}^*) \leq C(\boldsymbol{f})$ for all $\boldsymbol{f} \in \boldsymbol{F}$. $\square$

With the bounds of $C(\boldsymbol{f}^\kappa)$ given in Lemma 5, the real question is if $\zeta(\kappa, n)$ is a tight bound. Since $\zeta(\kappa, n)$ is obtained from the comparison between $C(\boldsymbol{f}^\kappa)$ and $C(\boldsymbol{f}^*)$, it is unclear if there is a case when we indeed have $C(\boldsymbol{f}^\kappa) = \zeta(\kappa, n) C(\boldsymbol{f}^0)$. We know for sure that $\zeta(\kappa, n)$ is not tight when κ is small, since $\zeta(0, n)$ is not equal to 1. We provide a partial answer for this question.

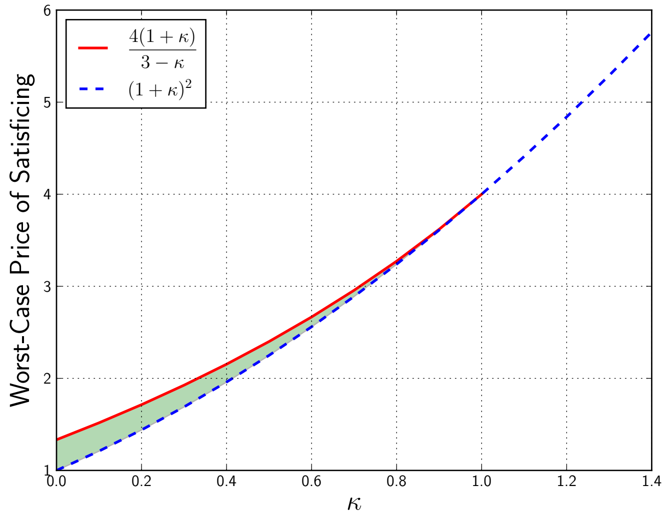


Figure 1: The worst-case price of satisficing with linear travel time functions. Note that when $n = 1$, the right-hand-side of (23) becomes $\frac{4(1+\kappa)}{3-\kappa}$. When $\kappa \geq 1$, we know for sure that the worst-case price of satisficing is exactly the dotted line. When $\kappa \leq 1$, the worst case falls in the shaded interval between the solid line and dotted line.

In Lemma 3 of Christodoulou et al. (2011), the existence of a network instance with $C(\boldsymbol{f}^\kappa) = (1+\kappa)^{n+1} C(\boldsymbol{f}^*)$ is shown for $\kappa \geq (n+1)^{1/n} - 1$ via an example. The same example is, however, valid for all $\kappa \geq 0$. Note that, in the same example, the system optimal flow is also at user equilibrium. Therefore, the worst-case PoSat is at least $(1+\kappa)^{n+1}$ for all $\kappa \geq 0$. Thus, we obtain the following theorem:

Theorem 2. *When $0 \leq \kappa \leq (n+1)^{1/n} - 1$, the worst-case PoSat falls in the following interval:*

$$(1+\kappa)^{n+1} \leq \sup_{\rho} \mathsf{PoSat}(\rho) \leq \left(\frac{1}{1+\kappa} - \frac{n}{(n+1)^{(n+1)/n}} \right)^{-1}. \tag{23}$$

When $\kappa \geq (n+1)^{1/n} - 1$, the worst-case PoSat is exactly:

$$\sup_{\rho} \mathsf{PoSat}(\rho) = (1+\kappa)^{n+1}. \tag{24}$$

Figure 1 shows the bounds in Theorem 2 for the linear travel time function case. For smaller κ values, the worst-case PoSat falls in the shaded interval, while for larger κ values, it is exactly $(1+\kappa)^2$. When κ is zero, we have $\boldsymbol{f}^\kappa = \boldsymbol{f}^0$; hence, we must have the PoSat approach to 1. With this observation, we naturally ask a question: Does $(1+\kappa)^{n+1}$ provide a tight bound on the worst-case PoSat for all values of $\kappa \geq 0$? We present partial answers to this question in the following sections.

4.2 Increased Travel Demands and Travel Time Functions

We first define new sets of flow vectors. When the travel demand Q_w for each $w \in \mathcal{W}$ is multiplied by the factor $1+\kappa$, we define

$$\begin{aligned}
\boldsymbol{F}_{1+\kappa} &= \left\{ \boldsymbol{f} : \sum_{p \in \mathcal{P}_w} f_p = (1+\kappa) Q_w \quad \forall w \in \mathcal{W}, \qquad f_p \geq 0 \quad \forall p \in \mathcal{P} \right\}, \\
\boldsymbol{V}_{1+\kappa} &= \left\{ \boldsymbol{v} : v_a = \sum_{p \in \mathcal{P}} \delta_a^p f_p \quad \forall a \in \mathcal{A}, \qquad \boldsymbol{f} \in \boldsymbol{F}_{1+\kappa} \right\}, \\
\boldsymbol{X}_{1+\kappa} &= \left\{ \boldsymbol{x} : x_a^w = \sum_{p \in \mathcal{P}_w} \delta_a^p f_p \quad \forall a \in \mathcal{A}, w \in \mathcal{W} \qquad \boldsymbol{f} \in \boldsymbol{F}_{1+\kappa} \right\}.
\end{aligned}$$

The above three sets can equivalently be written as follows:

$$\begin{aligned}
\boldsymbol{F}_{1+\kappa} &= \{(1+\kappa)\boldsymbol{f} : \boldsymbol{f} \in \boldsymbol{F}\}, \\
\boldsymbol{V}_{1+\kappa} &= \{(1+\kappa)\boldsymbol{v} : \boldsymbol{v} \in \boldsymbol{V}\}, \\
\boldsymbol{X}_{1+\kappa} &= \{(1+\kappa)\boldsymbol{x} : \boldsymbol{x} \in \boldsymbol{X}\}.
\end{aligned}$$

We will use 'hat' for flow vectors in these sets, for example, $\widehat{\boldsymbol{f}}^\kappa \in \boldsymbol{F}_{1+\kappa}$, while without hat in the original sets as in $\boldsymbol{f}^\kappa \in \boldsymbol{F}$.

We consider cases when the travel time functions $t_a(\cdot)$ are polynomials of order n, in particular, the following form of *asymmetric* arc travel time function for each $a \in \mathcal{A}$:

$$\begin{aligned}
t_a(\boldsymbol{v}) &= \sum_{m=0}^{n} b_{am} \left(\sum_{e \in \mathcal{A}} d_{aem} v_e \right)^m \\
&= \sum_{m=0}^{n} b_{am} \left(\boldsymbol{d}_{am}^\top \boldsymbol{v} \right)^m
\end{aligned} \tag{25}$$

for some constants b_{am} for $m = 0, 1, ..., n$ and d_{aem} for $e \in \mathcal{A}$ and $m = 0, 1, ..., n$. Note that we use the vector form $\boldsymbol{d}_{am} = (d_{aem} : e \in \mathcal{A})$. The travel time function (25) is a general form of the travel time functions considered in the traffic equilibrium literature (Meng et al., 2014; Panicucci et al., 2007). If $\boldsymbol{d}_{am}$ is a unit vector such that d_{aem} is 1 if $a = e$ and 0 otherwise, we have a separable polynomial arc travel time function that has been used in the literature popularly (Christodoulou et al., 2011; Roughgarden and Tardos, 2002):

$$t_a(v_a) = \sum_{m=0}^{n} b_{am} (v_a)^m = b_{a0} + b_{a1} v_a + b_{a2} (v_a)^2 + \cdots + b_{an} (v_a)^n. \tag{26}$$

Lemma 6. *With the polynomial travel time function* (25)*, for any* $\boldsymbol{f} \in \boldsymbol{F}$*, we have*

$$C((1+\kappa)\boldsymbol{f}) \leq (1+\kappa)^{n+1} C(\boldsymbol{f}) \tag{27}$$

for all $\kappa \geq 0$ and $n \geq 0$.

Proof of Lemma 6. By simple comparison, we can show

$$\begin{aligned}
C((1+\kappa)\boldsymbol{f}) = Z((1+\kappa)\boldsymbol{v}) &= \sum_{a\in\mathcal{A}} \left(\sum_{m=0}^{n} b_{am} \Big((1+\kappa) \boldsymbol{d}_{am}^{\top} \boldsymbol{v} \Big)^{m} \right) (1+\kappa) v_a \\
&\leq (1+\kappa)^{n+1} \left(\sum_{m=0}^{n} b_{am} \Big(\boldsymbol{d}_{am}^{\top} \boldsymbol{v} \Big)^{m} \right) v_a \\
&= (1+\kappa)^{n+1} Z(\boldsymbol{v}) \\
&= (1+\kappa)^{n+1} C(\boldsymbol{f})
\end{aligned}$$

where $\boldsymbol{v}$ is the arc flow vector from $\boldsymbol{f} \mapsto \boldsymbol{v}$. □

4.3 Cases with Separable, Monomial Link Travel Time Functions

As a simple case, we consider separable, monomial functions of degree n for link travel time of the following form:

$$t_a(v_a) = b_a(v_a)^n \tag{28}$$

with a positive scalar b_a for each $a \in \mathcal{A}$ and nonnegative constant n.

It is well known (Beckmann et al., 1956) that $\boldsymbol{v}^0 \in \boldsymbol{F}$ is a user equilibrium flow, if and only if it minimizes the following potential function

$$\Phi(\boldsymbol{v}) = \sum_{a\in\mathcal{A}} \int_0^{v_a} t_a(u)\,\mathrm{d}u = \sum_{a\in\mathcal{A}} \frac{b_a}{n+1} (v_a)^{n+1}$$

when the link travel time functions are separable, so that the integral is well defined. Similarly, $\boldsymbol{v}^\kappa \in \boldsymbol{V}$ is a κ-MSatUE flow, if it is a solution to UE-PE-$\boldsymbol{V}$, or equivalently, if it minimizes the following potential function (Christodoulou et al., 2011)

$$\Psi(\boldsymbol{v};\boldsymbol{\lambda}) = \sum_{a\in\mathcal{A}} \int_0^{v_a} \lambda_a t_a(u)\,\mathrm{d}u = \sum_{a\in\mathcal{A}} \frac{\lambda_a b_a}{n+1} (v_a)^{n+1}$$

for some $\lambda_a \in [\frac{1}{1+\kappa}, 1]$ for each $a \in \mathcal{A}$.

When travel time functions are separable, we can show the following result (Englert et al., 2010; Takalloo and Kwon, 2018):

Lemma 7. *When the link travel time functions are in the form of (26), let $\boldsymbol{f}^0 \in \boldsymbol{F}$ and $\widehat{\boldsymbol{f}}^0 \in \boldsymbol{F}_{1+\kappa}$ be the UE flows with the corresponding travel demands. We can show*

$$C(\widehat{\boldsymbol{f}}^0) \leq (1+\kappa)^{n+1} C(\boldsymbol{f}^0) \tag{29}$$

for all $\kappa \geq 0$ and $n \geq 0$.

Although Englert et al. (2010) consider cases with a single OD pair only with interest in the changes in the path travel time, the same technique can be used to prove Lemma 7 for cases with multiple OD pairs.

Theorem 3. *When the link travel time functions are of the form (28), let $\boldsymbol{f}^\kappa \in \boldsymbol{F}$ be a solution to* $\mathsf{UE\text{-}PE\text{-}}\boldsymbol{V}$ *with $\boldsymbol{v} \xrightarrow{\text{any}} \boldsymbol{f}$ and $\boldsymbol{f}^0 \in \boldsymbol{F}_{1+\kappa}$ be the UE flows. Then we have $C(\boldsymbol{f}^\kappa) \leq C(\widehat{\boldsymbol{f}}^0)$, and consequently $C(\boldsymbol{f}^\kappa) \leq (1+\kappa)^{n+1} C(\boldsymbol{f}^0)$ for all $\kappa \geq 0$.*

Proof of Theorem 3. Since $\widehat{\boldsymbol{v}}^0 \in \boldsymbol{F}_{1+\kappa}$ is an user equilibrium flow that minimizes $\Phi(\boldsymbol{v})$, we have

$$\Phi(\widehat{\boldsymbol{v}}^0) \leq \Phi\big((1+\kappa)\boldsymbol{v}^\kappa\big),$$

which implies

$$\sum_{a\in\mathcal{A}} \frac{b_a(\widehat{v}_a^0)^{n+1}}{n+1} \leq \sum_{a\in\mathcal{A}} \frac{b_a\big((1+\kappa)v_a^\kappa\big)^{n+1}}{n+1} = (1+\kappa)^{n+1} \sum_{a\in\mathcal{A}} \frac{b_a(v_a^\kappa)^{n+1}}{n+1}. \tag{30}$$

Since $\boldsymbol{v}^\kappa \in \boldsymbol{V}$ is a solution to $\mathsf{UE\text{-}PE\text{-}}\boldsymbol{V}$, we have

$$\Psi(\boldsymbol{v}^{\boldsymbol{\kappa}}; \boldsymbol{\lambda}) \leq \Phi\Big(\frac{\widehat{\boldsymbol{v}}^0}{1+\kappa}; \boldsymbol{\lambda}\Big),$$

for some $\boldsymbol{\lambda}$. Therefore, we have

$$\sum_{a\in\mathcal{A}} \frac{\lambda_a b_a(v_a^\kappa)^{n+1}}{n+1} \leq \sum_{a\in\mathcal{A}} \frac{\lambda_a b_a(v_a^0)^{n+1}}{(n+1)(1+\kappa)^{n+1}} = \frac{1}{(1+\kappa)^{n+1}} \sum_{a\in\mathcal{A}} \frac{\lambda_a b_a(\widehat{v}_a^0)^{n+1}}{n+1}.$$

Since $\lambda_a \in [\frac{1}{1+\kappa}, 1]$, we obtain

$$\frac{1}{1+\kappa} \sum_{a\in\mathcal{A}} \frac{b_a(v_a^\kappa)^{n+1}}{n+1} \leq \frac{1}{(1+\kappa)^{n+1}} \sum_{a\in\mathcal{A}} \frac{b_a(\widehat{v}_a^0)^{n+1}}{n+1}$$

which implies

$$(1+\kappa)^n \sum_{a\in\mathcal{A}} \frac{b_a(v_a^\kappa)^{n+1}}{n+1} \leq \sum_{a\in\mathcal{A}} \frac{b_a(\widehat{v}_a^0)^{n+1}}{n+1} \tag{31}$$

Let us assume that $C(\boldsymbol{f}^\kappa) > C(\widehat{\boldsymbol{f}}^0)$, which is equivalent to

$$\sum_{a\in\mathcal{A}} b_a(\widehat{v}_a^0)^{n+1} < \sum_{a\in\mathcal{A}} b_a(v_a^\kappa)^{n+1} \tag{32}$$

From $A \times (30) + B \times (31) + C \times (32)$ for any positive constants A, B and C, we obtain

$$\theta_1 \sum_{a\in\mathcal{A}} \frac{b_a(\widehat{v}_a^0)^{n+1}}{n+1} < \theta_2 \sum_{a\in\mathcal{A}} \frac{b_a(v_a^\kappa)^{n+1}}{n+1} \tag{33}$$

where

$$
\begin{aligned}
\theta_1 &= A - B + C(n+1) \\
\theta_2 &= A(1+\kappa)^{n+1} - B(1+\kappa)^n + C(n+1).
\end{aligned}
$$

In particular, consider A, B and C as follows:

$$
\begin{aligned}
A &= (n+1)\big((1+\kappa)^n - 1\big) \\
B &= (n+1)\big((1+\kappa)^n - 1\big) + (n+1)\kappa(1+\kappa)^{n+1} \\
C &= \kappa(1+\kappa)^{n+1}
\end{aligned}
$$

We observe that A, B and C are all positive and $\theta_1 = 0$. We also see that

$$
\theta_2 = -(n+1)\kappa^2(1+\kappa)^n\big((1+\kappa)^{n+1} - 1\big) \leq 0
$$

for all $\kappa \geq 0$ and $n \geq 0$, which leads to a contradiction. Therefore, we have

$$
C(\boldsymbol{f}^\kappa) \leq C(\widehat{\boldsymbol{f}}^0),
$$

and from Lemma 7, we obtain

$$
C(\boldsymbol{f}^\kappa) \leq C(\widehat{\boldsymbol{f}}^0) \leq (1+\kappa)^{n+1} C(\boldsymbol{f}^0),
$$

which completes the proof. □

Note that the bound obtained in Theorem 3 relies on the sufficient condition, not a necessary condition. Therefore, the result is not applicable to all MSatUE flows, although it provides a useful bound in the framework of UE-PE models.

4.4 Cases with Separable Link Travel Time Functions

We consider general polynomial, separable link travel functions in the form of (26).

Theorem 4. *Suppose that the link travel time functions are in the form of* (26)*. Let* $\boldsymbol{f}^\kappa \in \boldsymbol{F}$ *be any* κ*-MSatUE and* $\widehat{\boldsymbol{f}}^0 \in \boldsymbol{F}_{1+\kappa}$ *be the UE flow. Suppose that* $\kappa \geq 0$ *is sufficiently small, in particular, so that*

$$
\sum_{p\in\mathcal{P}} [c_p(\widehat{\boldsymbol{f}}^0) - c_p(\boldsymbol{f}^\kappa)](\widehat{f}_p^0 - f_p^\kappa) \geq \kappa \sum_{p\in\mathcal{P}} c_p(\boldsymbol{f}^\kappa) \left| \widehat{f}_p^0 - f_p^\kappa \right|. \tag{34}
$$

Then we have $C(\boldsymbol{f}^\kappa) \leq C(\widehat{\boldsymbol{f}}^0)$*. Consequently* $C(\boldsymbol{f}^\kappa) \leq (1+\kappa)^{n+1} C(\boldsymbol{f}^0)$*, and* $\sup_\rho \mathsf{PoSat}(\rho) = (1+\kappa)^{n+1}$*.*

Proof of Theorem 4. By slightly modifying the proof of Theorem 5, we can show $C(\boldsymbol{f}^\kappa) \leq C(\widehat{\boldsymbol{f}}^0)$. By Lemmas 6 and 7, we complete the proof. □

Theorem 4 depends on condition (34) and a similar condition appears in general asymmetric cases as in Theorem 5. We discuss this condition in Section 4.6.

4.5 General Cases with Asymmetric Link Travel Time Functions

We consider asymmetric link travel time functions (25), in which case Lemma 7 is not applicable. We first observe that the multiple of a PRUE flow, $(1+\kappa)\boldsymbol{f}^0$, provides a satisficing solution to the traffic equilibrium problem with the increased travel demand.

Lemma 8. *Suppose $t_a(\cdot)$ are polynomials of order n. If $\boldsymbol{f}^0 \in \boldsymbol{F}$ is a PRUE flow, then $(1+\kappa)\boldsymbol{f}^0$ is a σ-MSatUE flow with $\sigma = (1+\kappa)^n - 1$ in $\boldsymbol{F}_{1+\kappa}$. When $n = 1$, we have $\sigma = \kappa$.*

Proof. Let $\bar{\boldsymbol{f}} = (1+\kappa)\boldsymbol{f}^0$, and $\bar{\boldsymbol{v}} = (1+\kappa)\boldsymbol{v}^0$ for the corresponding arc flow vectors. If the condition

$$\sum_{a \in \mathcal{A}} \left(\sum_{m=0}^{n} \lambda_{am} b_{am} \left(\boldsymbol{d}_{am}^\top \bar{\boldsymbol{v}} \right)^m \right) (v'_a - \bar{v}_a) \geq 0 \qquad \forall \boldsymbol{v}' \in \boldsymbol{V}_{1+\kappa} \tag{35}$$

holds for some constants $\lambda_{am} \in [\frac{1}{1+\sigma}, 1]$ for $m = 0, 1, ..., n$ and $a \in \mathcal{A}$, then we can find $\lambda_a \in [\frac{1}{1+\sigma}, 1]$ such that

$$\lambda_a \sum_{m=0}^{n} b_{am} \left(\boldsymbol{d}_{am}^\top \bar{\boldsymbol{v}} \right)^m = \sum_{m=0}^{n} \lambda_{am} b_{am} \left(\boldsymbol{d}_{am}^\top \bar{\boldsymbol{v}} \right)^m$$

for all $a \in \mathcal{A}$; consequently, by Theorem 1, $\bar{\boldsymbol{f}}$ is a σ-MSatUE flow in $\boldsymbol{F}_{1+\kappa}$.

Since $\boldsymbol{v}^0$ is PRUE for $\boldsymbol{V}$, we know that

$$\sum_{a \in \mathcal{A}} \left(\sum_{m=0}^{n} b_{am} \left(\boldsymbol{d}_{am}^\top \boldsymbol{v}^0 \right)^m \right) (v_a - v_a^0) \geq 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

Therefore

$$\sum_{a \in \mathcal{A}} \left(\sum_{m=0}^{n} \frac{1}{(1+\kappa)^m} b_{am} \left((1+\kappa) \boldsymbol{d}_{am}^\top \boldsymbol{v}^0 \right)^m \right) ((1+\kappa)v_a - (1+\kappa)v_a^0) \geq 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

Letting for all $a \in \mathcal{A}$

$$\begin{aligned} \lambda_{am} &= \frac{1}{(1+\kappa)^m}, \qquad m = 0, 1, ..., n \\ \bar{v}_a &= (1+\kappa)v_a^0, \\ v'_a &= (1+\kappa)v_a, \end{aligned}$$

we observe that $\lambda_{am} \in [\frac{1}{1+\sigma}, 1]$ and we obtain (35); hence proof. $\square$

By introducing an additional condition, we compare MSatUE flows with the proportional travel demand increase, and obtain the worst-case bound of PoSat.

Theorem 5. *Let $\boldsymbol{f}^\kappa \in \boldsymbol{F}$ be any κ-MSatUE and $\widehat{\boldsymbol{f}}^\sigma \in \boldsymbol{F}_{1+\kappa}$ be any σ-MSatUE flows with the corresponding travel demands, when $\sigma = (1+\kappa)^n - 1$. Suppose that $\kappa \geq 0$ is sufficiently small, in particular, so that*

$$\sum_{p\in\mathcal{P}} [c_p(\widehat{\boldsymbol{f}}^\sigma) - c_p(\boldsymbol{f}^\kappa)](\widehat{f}_p^\sigma - f_p^\kappa) \geq \sigma \sum_{p\in\mathcal{P}} \max\{c_p(\widehat{\boldsymbol{f}}^\sigma), c_p(\boldsymbol{f}^\kappa)\} \left|\widehat{f}_p^\sigma - f_p^\kappa\right|. \tag{36}$$

Then we have $C(\boldsymbol{f}^\kappa) \leq C(\widehat{\boldsymbol{f}}^\sigma)$. Consequently $C(\boldsymbol{f}^\kappa) \leq (1+\kappa)^{n+1} C(\boldsymbol{f}^0)$, and $\sup_\rho \mathsf{PoSat}(\rho) = (1+\kappa)^{n+1}$.

Proof of Theorem 5. We decompose $\mathcal{P}_w$ for each OD pair w into the following four subsets:

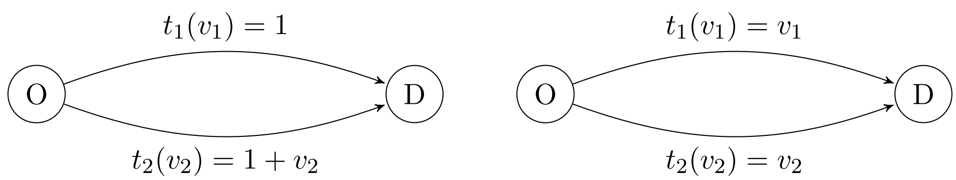
$$\begin{aligned}
\mathcal{P}_w^1 &= \{p \in \mathcal{P}_w : \widehat{f}_p^\sigma > 0,\ f_p^\kappa > 0,\ \widehat{f}_p^\sigma - f_p^\kappa \geq 0\}, \\
\mathcal{P}_w^2 &= \{p \in \mathcal{P}_w : \widehat{f}_p^\sigma > 0,\ f_p^\kappa > 0,\ \widehat{f}_p^\sigma - f_p^\kappa < 0\}, \\
\mathcal{P}_w^3 &= \{p \in \mathcal{P}_w : \widehat{f}_p^\sigma > 0,\ f_p^\kappa = 0\}, \\
\mathcal{P}_w^4 &= \{p \in \mathcal{P}_w : \widehat{f}_p^\sigma = 0,\ f_p^\kappa > 0\}.
\end{aligned}$$

We ignore cases with $\widehat{f}_p^\sigma = 0$ and $f_p^\kappa = 0$. Note that $\widehat{f}_p^\sigma - f_p^\kappa > 0$ for $p \in \mathcal{P}_w^3$ and $\widehat{f}_p^\sigma - f_p^\kappa < 0$ for $p \in \mathcal{P}_w^4$. From the definition of MSatUE flows, we have

$$\begin{aligned}
\widehat{f}_p^\sigma > 0 &\implies c_p(\widehat{\boldsymbol{f}}^\sigma) \leq (1+\sigma)\mu_w(\widehat{\boldsymbol{f}}^\sigma), \\
f_p^\kappa > 0 &\implies c_p(\boldsymbol{f}^\kappa) \leq (1+\kappa)\mu_w(\boldsymbol{f}^\kappa),
\end{aligned}$$

for all $p \in \mathcal{P}_w, w \in \mathcal{W}$. In addition, $\mu_w(\widehat{\boldsymbol{f}}^\sigma) \leq c_p(\widehat{\boldsymbol{f}}^\sigma)$ and $\mu_w(\boldsymbol{f}^\kappa) \leq c_p(\boldsymbol{f}^\kappa)$ for all $p \in \mathcal{P}$ by definition. Therefore, we have

$$\begin{aligned}
&\sum_{p\in\mathcal{P}} [c_p(\widehat{\boldsymbol{f}}^\sigma) - c_p(\boldsymbol{f}^\kappa)](\widehat{f}_p^\sigma - f_p^\kappa) \\
&\leq \sum_{w\in\mathcal{W}} \Bigg\{ \sum_{p\in\mathcal{P}_w^1} \Big[(1+\sigma)\mu_w(\widehat{\boldsymbol{f}}^\sigma) - \mu_w(\boldsymbol{f}^\kappa)\Big](\widehat{f}_p^\sigma - f_p^\kappa) + \sum_{p\in\mathcal{P}_w^2} \Big[\mu_w(\widehat{\boldsymbol{f}}^\sigma) - (1+\kappa)\mu_w(\boldsymbol{f}^\kappa)\Big](\widehat{f}_p^\sigma - f_p^\kappa) \\
&\qquad + \sum_{p\in\mathcal{P}_w^3} \Big[(1+\sigma)\mu_w(\widehat{\boldsymbol{f}}^\sigma) - \mu_w(\boldsymbol{f}^\kappa)\Big](\widehat{f}_p^\sigma - f_p^\kappa) + \sum_{p\in\mathcal{P}_w^4} \Big[\mu_w(\widehat{\boldsymbol{f}}^\sigma) - (1+\kappa)\mu_w(\boldsymbol{f}^\kappa)\Big](\widehat{f}_p^\sigma - f_p^\kappa) \Bigg\} \\
&= \sum_{w\in\mathcal{W}} \Bigg\{ \sum_{p\in\mathcal{P}_w} \Big[\mu_w(\widehat{\boldsymbol{f}}^\sigma) - \mu_w(\boldsymbol{f}^\kappa)\Big](\widehat{f}_p^\sigma - f_p^\kappa) + \sigma \sum_{p\in\mathcal{P}_w^1\cup\mathcal{P}_w^3} \mu_w(\widehat{\boldsymbol{f}}^\sigma)(\widehat{f}_p^\sigma - f_p^\kappa) \\
&\qquad\qquad - \kappa \sum_{p\in\mathcal{P}_w^2\cup\mathcal{P}_w^4} \mu_w(\boldsymbol{f}^\kappa)(\widehat{f}_p^\sigma - f_p^\kappa) \Bigg\} \\
&\leq \sum_{w\in\mathcal{W}} \Bigg\{ \sum_{p\in\mathcal{P}_w} \Big[\mu_w(\widehat{\boldsymbol{f}}^\sigma) - \mu_w(\boldsymbol{f}^\kappa)\Big](\widehat{f}_p^\sigma - f_p^\kappa) + \sigma \sum_{p\in\mathcal{P}_w} \max\{\mu_w(\widehat{\boldsymbol{f}}^\sigma), \mu_w(\boldsymbol{f}^\kappa)\} \left|\widehat{f}_p^\sigma - f_p^\kappa\right| \Bigg\} \\
&\leq \sum_{w\in\mathcal{W}} \sum_{p\in\mathcal{P}_w} \Big[\mu_w(\widehat{\boldsymbol{f}}^\sigma) - \mu_w(\boldsymbol{f}^\kappa)\Big](\widehat{f}_p^\sigma - f_p^\kappa) + \sigma \sum_{p\in\mathcal{P}} \max\{c_p(\widehat{\boldsymbol{f}}^\sigma), c_p(\boldsymbol{f}^\kappa)\} \left|\widehat{f}_p^\sigma - f_p^\kappa\right|.
\end{aligned}$$



(a) Example 1 (b) Example 2

Figure 2: Examples where the travel demand is Q from node O to node D.

From (36), we obtain

$$
\begin{aligned}
0 &\leq \sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}} \left[\mu_w(\widehat{\boldsymbol{f}}^\sigma) - \mu_w(\boldsymbol{f}^\kappa) \right] (\widehat{f}_p^\sigma - f_p^\kappa) \\
&= \sum_{w \in \mathcal{W}} \left[\mu_w(\widehat{\boldsymbol{f}}^\sigma) - \mu_w(\boldsymbol{f}^\kappa) \right] \left(\sum_{p \in \mathcal{P}} \widehat{f}_p^\sigma - \sum_{p \in \mathcal{P}} f_p^\kappa \right) \\
&= \sum_{w \in \mathcal{W}} \left[\mu_w(\widehat{\boldsymbol{f}}^\sigma) - \mu_w(\boldsymbol{f}^\kappa) \right] (\widehat{Q}_w - Q_w) \\
&= \kappa \sum_{w \in \mathcal{W}} \mu_w(\widehat{\boldsymbol{f}}^\sigma) Q_w - \kappa \sum_{w \in \mathcal{W}} \mu_w(\boldsymbol{f}^\kappa) Q_w \\
&= \frac{\kappa}{1+\kappa} \sum_{w \in \mathcal{W}} \mu_w(\widehat{\boldsymbol{f}}^\sigma) \widehat{Q}_w - \kappa \sum_{w \in \mathcal{W}} \mu_w(\boldsymbol{f}^\kappa) Q_w \\
&\leq \frac{\kappa}{1+\kappa} \sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}_w} c_p(\widehat{\boldsymbol{f}}^\sigma) \widehat{f}_p^\sigma - \frac{\kappa}{1+\kappa} \sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}_w} c_p(\boldsymbol{f}^\kappa) f_p^\kappa \\
&= \frac{\kappa}{1+\kappa} C(\widehat{\boldsymbol{f}}^\sigma) - \frac{\kappa}{1+\kappa} C(\boldsymbol{f}^\kappa).
\end{aligned}
$$

Lemmas 6 and 8 complete the proof. □

Note that condition (36) is stronger than condition (34) for separable travel time functions. This is natural, since we consider more general classes of travel time functions.

4.6 Illustrative Examples

For the illustration purpose, we consider two examples in Figure 2 with linear travel time functions, where $n = 1$. In Example 1, the travel time function in the first arc is not increasing. We can verify that

$$
\max C(\boldsymbol{f}^\kappa) = \begin{cases} Q + \kappa^2 & \text{if } \kappa \leq Q, \qquad \text{with } \boldsymbol{f}^\kappa = (Q - \kappa, \kappa) \\ (1+Q)Q & \text{if } \kappa \geq Q, \qquad \text{with } \boldsymbol{f}^\kappa = (0, Q) \end{cases}
$$

among all κ-MSatUE flows in $\boldsymbol{F}$ and

$$
C(\widehat{\boldsymbol{f}}^0) = (1+\kappa)Q \qquad \text{with } \widehat{\boldsymbol{f}}^0 = (1+\kappa)\boldsymbol{f}^0 = ((1+\kappa)Q, 0).
$$

among all κ-MSatUE flows in $\boldsymbol{F}_{1+\kappa}$. Comparing the two quantities, we observe $C(\boldsymbol{f}^\kappa) \leq C(\widehat{\boldsymbol{f}}^0)$ in both cases. To prove Theorem 5, condition (34) needs to hold only for these two flow vectors. Regardless of the value of κ, however, it is impossible to satisfy condition (34), although the worst-case PoSat bound $(1+\kappa)^{n+1}$ still holds for all $\kappa \geq 0$. The price of satisficing is $1 + \frac{\kappa^2}{Q}$ if $\kappa < Q$ and $1 + Q$ if $\kappa \geq Q$ in this example, both of which are less than $(1+\kappa)^2$.

On the other hand, in Example 2, we have strictly monotone travel time functions in both arcs. Similarly, we consider

$$\begin{aligned} \max C(\boldsymbol{v}^\kappa) &= \frac{2+2\kappa+\kappa^2}{(2+\kappa)^2} Q \qquad && \text{with } \boldsymbol{f}^\kappa = \left(\frac{Q}{2+\kappa}, \frac{(1+\kappa)Q}{2+\kappa} \right) \\ C(\widehat{\boldsymbol{v}}^0) &= \frac{(1+\kappa)^2}{2} Q && \text{with } \widehat{\boldsymbol{f}}^0 = (1+\kappa)\boldsymbol{f}^0 = \left(\frac{(1+\kappa)Q}{2}, \frac{(1+\kappa)Q}{2} \right) \end{aligned}$$

and can verify that $C(\boldsymbol{f}^\kappa) \leq C(\widehat{\boldsymbol{f}}^0)$ for all $\kappa \geq 0$. In Example 2, we note that (36) holds for $\kappa \leq 0.206$. In this example, we observe that the price of satisficing is $\frac{2(2+2\kappa+\kappa^2)}{(2+\kappa)^2}$, which is no greater than $(1+\kappa)^2$ for all $\kappa \geq 0$.

4.7 Other Approaches

When there is a single origin and multiple destinations, i.e., a single common origin node, in the network, Kleer and Schäfer (2016) introduces the notion of the *deviation ratio* that compares the system performances of the user equilibrium and the equilibrium with *deviated* travel time functions $\widetilde{t}_a(\cdot)$. The notion deviation may also be interpreted as perception in our definition. In a special case, the deviation ratio is reduced to the price of risk aversion (Nikolova and Stier-Moses, 2015) that compares the performances of equilibria among risk-averse and risk-neutral network users.

Kleer and Schäfer (2016) define the (separable) deviated travel time functions with the following bounds:

$$t_a(v_a) + \alpha t_a(v_a) \leq \widetilde{t}_a(v_a) \leq t_a(v_a) + \beta t_a(v_a) \tag{37}$$

where $-1 \leq \alpha \leq 0 \leq \beta$. The consideration of this deviated travel time function generalizes our UE-PE model where $\alpha = -\frac{\kappa}{1+\kappa}$ and $\beta = 0$. Kleer and Schäfer (2016) show that the worst-case deviation ratio with (37) is bounded by

$$1 + \frac{\beta - \alpha}{1+\alpha} \left\lceil \frac{|\mathcal{N}| - 1}{2} \right\rceil Q.$$

Therefore, we obtain the following theorem:

Theorem 6 (Kleer and Schäfer, 2016)**.** *Consider a directed graph with a single common origin node with the total travel demand Q and let $|\mathcal{N}|$ be the number of nodes. Then we have*

$$\frac{Z(\boldsymbol{v}^\kappa)}{Z(\boldsymbol{v}^0)} \leq 1 + \kappa \left\lceil \frac{|\mathcal{N}| - 1}{2} \right\rceil Q \tag{38}$$

where $\boldsymbol{v}^\kappa$ *is a solution to* $\mathsf{UE\text{-}PE\text{-}}\boldsymbol{V}$ *in* (18).

Note that Theorem 6 only covers a subset of the entire MSatUE flows, as it is limited to the solutions $\mathsf{UE\text{-}PE\text{-}}\boldsymbol{V}$ in (18) and is applicable to cases with a *single* common origin. When Theorem 6 is applied in the examples in Figure 2, the bound (38) becomes $1 + \kappa Q$.

5 Numerical Bounds

To quantify PoSat in typical traffic networks and compare it with the analytical bound in Theorem 5, we define the worst-case problem for the total system travel time under MSatUE as follows:

$$\begin{aligned} \max \quad & Z(\boldsymbol{v}^\kappa) = \sum_{a \in \mathcal{A}} z_a(v_a^\kappa) = \sum_{a \in \mathcal{A}} t_a(\boldsymbol{v}^\kappa) v_a^\kappa \\ \text{subject to} \quad & \boldsymbol{v}^\kappa \text{ is an } \underline{\text{MSatUE}} \text{ flow with } \kappa \end{aligned} \tag{39}$$

To quantify the benefit of satisficing, we can minimize the objective function, instead of maximizing. Since MSatUE involves path-based definition and formulation, (39) is numerically more challenging to solve. Instead, we replace MSatUE by UE-PE. We know that the UE-PE models provide a subset of MSatUE traffic flow patterns as seen in Theorem 1; hence by using UE-PE models, we will obtain suboptimal solutions to (39).

Using $\mathsf{UE\text{-}PE\text{-}}\boldsymbol{X}$ in (13), we formulate the worst-case problem as follows:

$$\max \quad Z(\boldsymbol{v}^\kappa) = \sum_{a \in \mathcal{A}} z_a(v_a^\kappa) = \sum_{a \in \mathcal{A}} t_a(\boldsymbol{v}^\kappa) v_a^\kappa \tag{40}$$

$$\text{subject to} \quad \sum_{a \in \mathcal{A}} \sum_{w \in \mathcal{W}} (t_a(\boldsymbol{v}^\kappa) - \varepsilon_a^w)(x_a^w - x_a^{\kappa,w}) \geq 0 \qquad \forall \boldsymbol{x} \in \boldsymbol{X} \tag{41}$$

$$v_a^\kappa = \sum_{w \in \mathcal{W}} x_a^{\kappa,w} \qquad \forall a \in \mathcal{A} \tag{42}$$

$$0 \leq \varepsilon_a^w \leq \frac{\kappa}{1+\kappa} t_a(\boldsymbol{v}^\kappa) \qquad \forall a \in \mathcal{A} \tag{43}$$

Problem (40) is an instance of mathematical programs with equilibrium constraints (MPEC). We can replace the equilibrium condition (41) by the following KKT conditions to create a single-level optimization problem:

$$t_a(\boldsymbol{v}^\kappa) - \varepsilon_a^w + \pi_i^w - \pi_j^w \geq 0 \qquad \forall w \in \mathcal{W}, a \in \mathcal{A} \tag{44}$$

$$x_a^{\kappa,w}(t_a(\boldsymbol{v}^\kappa) - \varepsilon_a^w + \pi_i^w - \pi_j^w) = 0 \qquad \forall w \in \mathcal{W}, a \in \mathcal{A} \tag{45}$$

$$\sum_{a \in \mathcal{A}_i^+} x_a^{\kappa,w} - \sum_{a \in \mathcal{A}_i^-} x_a^{\kappa,w} = q_i^w \qquad \forall w \in \mathcal{W}, i \in \mathcal{N} \tag{46}$$

The resulting problem is a mathematical program with complementarity conditions (MPCC), which is nonlinear and nonconvex. Finding a global solution to MPCC problems is in general difficult, and Kleer and Schäfer (2016) has shown that solving the above MPCC optimally is NP-hard. We note that the problem is closely related to the worst-case problem proposed by Lou et al. (2010). While MPCC-based algorithms, as suggested by Lou et al. (2010), are valid for solving the MPCC problem, our intention is to use the sensitivity analysis for parametric variational inequalities by taking advantage of the form UE-PE-$\boldsymbol{V}$ in (18).

5.1 Sensitivity Analysis in $\boldsymbol{V}$

Letting $\widetilde{t}_a(\boldsymbol{v}, \lambda_a) = \lambda_a t_a(\boldsymbol{v})$, we consider, in vector form, the following parametric variational inequality problem:

$$\widetilde{\boldsymbol{t}}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\lambda}})^\top (\boldsymbol{v} - \bar{\boldsymbol{v}}) \geq 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V} \tag{47}$$

for any given $\bar{\boldsymbol{\lambda}}$. With the current parameter $\bar{\boldsymbol{\lambda}}$, the solution to (47) is $\bar{\boldsymbol{v}}$—and $\bar{\boldsymbol{f}}$. As the parameter changes from $\bar{\boldsymbol{\lambda}}$ to $\bar{\boldsymbol{\lambda}} + \boldsymbol{\lambda}'$, the flow vectors $\bar{\boldsymbol{v}}$ and $\bar{\boldsymbol{f}}$ will change to $\bar{\boldsymbol{v}} + \boldsymbol{v}'$ and $\bar{\boldsymbol{f}} + \boldsymbol{f}'$, respectively. We want to estimate $\boldsymbol{v}'$ and $\boldsymbol{f}'$, which will be used eventually to estimate $\frac{\partial Z}{\partial \lambda_a}$.

Sensitivity analysis and solution differentiability for variational inequalities have been studied in the literature, e.g., (Pang, 1990; Bonnans and Shapiro, 2000; Qiu and Magnanti, 1989). There are several methods available to estimate the directional derivative of $\boldsymbol{v}$ with respect to $\boldsymbol{\lambda}$ at $(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\lambda}})$ in direction of $\boldsymbol{\lambda}'$. We emphasize that any such method can be applied in our parametric variational inequality presentation (47) for the UE-PE problem, which is a salient feature of our UE-PE formulation. In this paper, we focus on the method by Patriksson (2004) to develop a computational method based on directional derivatives to quantify PoSat numerically.

Patriksson (2004) shows that $\boldsymbol{v}'$ may be characterized by the following system:

$$\boldsymbol{r}(\boldsymbol{v}', \boldsymbol{\lambda}')^\top (\boldsymbol{u} - \boldsymbol{v}') \geq 0 \qquad \forall \boldsymbol{u} \in \boldsymbol{K} \tag{48}$$

where we define that

$$\boldsymbol{r}(\boldsymbol{v}', \boldsymbol{\lambda}') = \nabla_{\boldsymbol{v}} \widetilde{\boldsymbol{t}}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\lambda}}) \boldsymbol{v}' + \nabla_{\boldsymbol{\lambda}} \widetilde{\boldsymbol{t}}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\lambda}}) \boldsymbol{\lambda}' \tag{49}$$

$$\boldsymbol{K}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\lambda}}) = \boldsymbol{T}_{\boldsymbol{V}}(\bar{\boldsymbol{v}}) \cap \widetilde{\boldsymbol{t}}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\lambda}})^\perp \tag{50}$$

$$\widetilde{\boldsymbol{t}}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\lambda}})^\perp = \left\{ \boldsymbol{y} : \boldsymbol{y}^\top \boldsymbol{t}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\lambda}}) = 0 \right\} \tag{51}$$

$$\boldsymbol{T}_{\boldsymbol{V}}(\bar{\boldsymbol{v}}) \text{ is the tangent cone of the set } \boldsymbol{V} \text{ at } \bar{\boldsymbol{v}}, \tag{52}$$

where $\nabla_{\boldsymbol{v}} \widetilde{\boldsymbol{t}}(\cdot, \cdot)$ and $\nabla_{\boldsymbol{\lambda}} \widetilde{\boldsymbol{t}}(\cdot, \cdot)$ represent the Jacobian of $\widetilde{\boldsymbol{t}}(\cdot, \cdot)$ with respect to $\boldsymbol{v}$ and $\boldsymbol{\lambda}$, respectively.

Patriksson (2004) shows that $\boldsymbol{K}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\lambda}})$ can be equivalently written as the set of $\boldsymbol{v}'$ such that:

$$v'_a = \sum_{p \in \mathcal{P}} \delta_a^p f'_p \qquad \forall a \in \mathcal{A}$$

$$\sum_{p \in \mathcal{P}_w} f'_p = 0 \qquad \forall w \in \mathcal{W}$$

$$f'_p \begin{cases} \text{free}, & \text{if } \bar{f}^\kappa_p > 0 \\ \geq 0, & \text{if } \bar{f}^\kappa_p = 0,\ \widetilde{c}_p(\bar{\boldsymbol{f}}, \bar{\boldsymbol{\lambda}}) = \widetilde{\mu}_w(\bar{\boldsymbol{f}}, \bar{\boldsymbol{\lambda}}) \\ = 0, & \text{if } \bar{f}^\kappa_p = 0,\ \widetilde{c}_p(\bar{\boldsymbol{f}}, \bar{\boldsymbol{\lambda}}) > \widetilde{\mu}_w(\bar{\boldsymbol{f}}, \bar{\boldsymbol{\lambda}}). \end{cases} \qquad \forall p \in \mathcal{P}_w, w \in \mathcal{W}$$

where path travel time functions $\widetilde{c}_p(\cdot,\cdot)$ and $\widetilde{\mu}(\cdot,\cdot)$ are defined corresponding to arc travel time functions $\widetilde{t}_a(\cdot,\cdot)$. Since the above representation requires path enumeration, we represent $\boldsymbol{K}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\lambda}})$ as the set of $\boldsymbol{v}'$ such that:

$$v'_a = \sum_{w \in \mathcal{W}} x'^w_a \qquad \forall a \in \mathcal{A} \tag{53}$$

$$\sum_{a \in \mathcal{A}^+_i} x'^w_a - \sum_{a \in \mathcal{A}^-_i} x'^w_a = 0 \qquad \forall i \in \mathcal{N},\ \forall w \in \mathcal{W} \tag{54}$$

$$x'^w_a \begin{cases} \text{free}\ , & \text{if } \bar{x}^w_a > 0 \\ \geq 0 & \text{if } \bar{x}^w_a = 0 \end{cases} \qquad \forall w \in \mathcal{W},\ i \in \mathcal{N} \tag{55}$$

$$\sum_{a \in \mathcal{A}} \widetilde{t}_a(\bar{\boldsymbol{v}}, \bar{\lambda}_a) v'_a = 0 \tag{56}$$

where (53)–(55) represent the tangent cone $\boldsymbol{T_V}(\bar{\boldsymbol{v}})$ and (56) represents the orthogonal space (51).

If $\nabla_{\boldsymbol{v}} \widetilde{\boldsymbol{t}}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\lambda}})$ is positive definite, then $\boldsymbol{r}(\cdot,\cdot)$ is strictly monotone. With the strict monotonicity of $\boldsymbol{r}(\cdot,\cdot)$, the solution $\boldsymbol{v}'$ to (48) is unique, and it provides the directional derivative of $\boldsymbol{v}$ with respect to $\boldsymbol{\lambda}$ at $(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\lambda}})$ in direction of $\boldsymbol{\lambda}'$. When link cost function is separable, we may obtain the solution to (48) by solving the following equivalent convex quadratic optimization problem (Josefsson and Patriksson, 2007):

$$\min_{\boldsymbol{v}'} \quad [\nabla_{\boldsymbol{v}} \widetilde{\boldsymbol{t}}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\lambda}}) \boldsymbol{\lambda}']^\top \boldsymbol{v}' + \frac{1}{2} \sum_{a \in \mathcal{A}} \frac{\partial \widetilde{t}_a(\bar{v}_a, \bar{\lambda}_a)}{\partial v_a} (v'_a)^2 \tag{57}$$

subject to (53)–(56)

for any given direction $\boldsymbol{\lambda}'$. When link cost functions are asymmetric, we cannot use problem (57) to obtain the solutions to problem (48). In this case, we can solve the variational inequality problem (48) by using a projection-based method such as the extragradient method that converge for monotone variational inequality problems (Facchinei and Pang, 2007).

5.2 A Gradient Projection Algorithm

We replace the MSatUE requirement in (39) by UE-PE-$\boldsymbol{V}$ in (18). Note that UE-PE-$\boldsymbol{F}$ and UE-PE-$\boldsymbol{X}$ may not be used in algorithms based on sensitivity analysis, since traffic equilibria in $\boldsymbol{F}$ or $\boldsymbol{X}$ are not unique in general; hence cannot guarantee the uniqueness of directional derivatives. With strict monotonicity of arc travel time functions $t_a(\cdot)$, we can assure the uniqueness as discussed in the above.

We consider

$$\begin{aligned}
\max_{\boldsymbol{\lambda}} \quad & Z = \sum_{a\in\mathcal{A}} t_a(\boldsymbol{v}^\kappa) v_a^\kappa && && (58)\\
\text{subject to} \quad & \sum_{a\in\mathcal{A}} \lambda_a t_a(\boldsymbol{v}^\kappa)(v_a - v_a^\kappa) \geq 0 && \forall \boldsymbol{v} \in \boldsymbol{V} && (59)\\
& \frac{1}{1+\kappa} \leq \lambda_a \leq 1 && \forall a \in \mathcal{A} && (60)\\
& \boldsymbol{v}^\kappa \in \boldsymbol{V} && && (61)
\end{aligned}$$

To approximate $\frac{\partial \boldsymbol{v}}{\partial \lambda_a}$, we consider arc $a \in \mathcal{A}$ and we construct a direction $\boldsymbol{\lambda}'$ with each element being

$$\lambda'_e = \begin{cases} 1 & \text{if } e = a \\ 0 & \text{if } e \neq a, e \in \mathcal{A} \end{cases}$$

In separable link cost function networks, given this direction $\boldsymbol{\lambda}'$, we solve (57) to obtain $\boldsymbol{v}'$, which approximates $\frac{\partial \boldsymbol{v}^\kappa}{\partial \lambda_a}$. For asymmetric link cost functions networks, we replace $\boldsymbol{\lambda}'$ in (48) and solve the equilibrium problem using any algorithm that can handle asymmetric link cost functions. We repeat this procedure for all $a \in \mathcal{A}$.

Once we obtained the Jacobian matrix $\nabla_{\boldsymbol{\lambda}} \boldsymbol{v}$, we compute the gradient as follows:

$$\nabla_{\boldsymbol{\lambda}} Z = \nabla_{\boldsymbol{\lambda}} \boldsymbol{v} \nabla_{\boldsymbol{v}} Z.$$

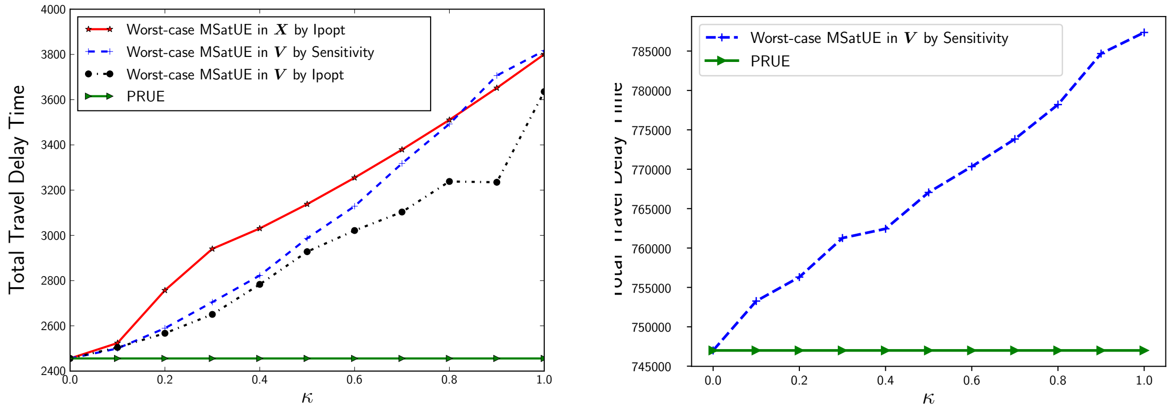
When the incumbent solution is $\bar{\boldsymbol{\lambda}}$ and its corresponding equilibrium solution is $\bar{\boldsymbol{v}}$, we have the following projection rule to solve (48):

$$\boldsymbol{\lambda}^{\text{new}} = \left[\bar{\boldsymbol{\lambda}} + \theta \nabla_{\boldsymbol{\lambda}} Z \Big|_{\boldsymbol{\lambda}=\bar{\boldsymbol{\lambda}}, \boldsymbol{v}=\bar{\boldsymbol{v}}} \right]_{\frac{1}{1+\kappa}}^{1}$$

with an appropriate step size θ, where $[x]_a^b$ is the projection on to the interval $[a, b]$, i.e. $\max(a, \min(x, b))$.

5.3 Numerical Experiments

In this section we present some examples to compare the total travel times in MSatUE and PRUE numerically for both separable and asymmetric networks. The benchmarks are solutions to (39) defined with UE-PE-$\boldsymbol{X}$ and UE-PE-$\boldsymbol{V}$, solved by the Ipopt nonlinear solver (Wächter and Biegler, 2006), after reformulating as a single-level optimization problem using KKT conditions. We use the Julia Language and the JuMP package (Dunning et al., 2017) for modeling and interfacing with the Ipopt solver.



(a) The nine-node network of Hearn and Ramana (1998)

(b) The Sioux Falls network

Figure 3: Comparing total travel times with separable travel time functions

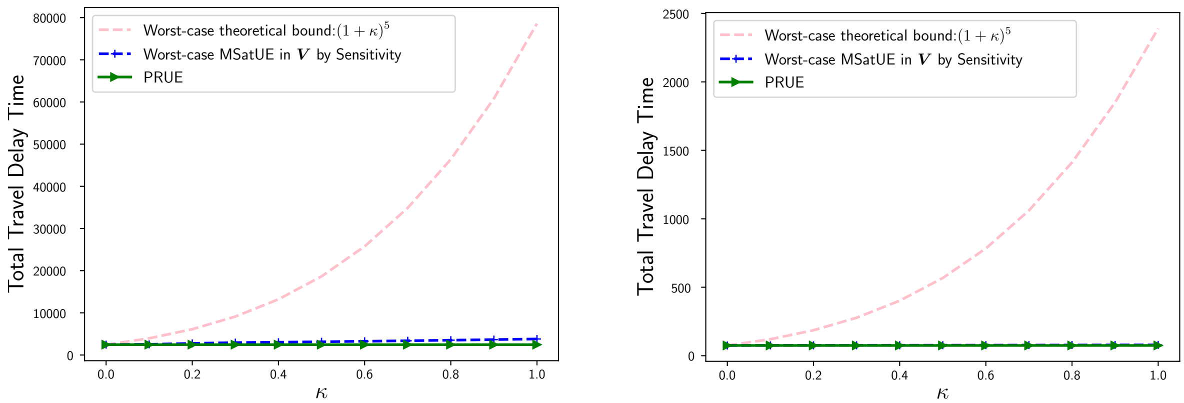
5.3.1 Separable Travel Time Function Networks

We present some examples to compare the total travel times in MSatUE and PRUE numerically and compare the numerical worst-cases with the analytical bound given in Theorem 4 for separable link cost function networks. We also test the performance of the sensitivity-based gradient projection algorithm for solving problem (58), which is the worst-case MSatUE problem with UE-PE-$\boldsymbol{V}$.

First, we consider the nine-node network presented in Hearn and Ramana (1998). The nine-node network consists of 9 nodes and 18 arcs, and the travel time functions are polynomials of order $n = 4$. The comparison result is presented in Figure 3a. As (39) is a non-convex problem, the Ipopt solver can produce a local minimum at best. The sensitivity-based algorithm certainly outperforms the Ipopt solver in $\boldsymbol{V}$, since the sensitivity-based algorithm utilizes the specific structure of parametric variational inequalities. On the other hand, the Ipopt solver in $\boldsymbol{X}$ outperforms the sensitivity-based algorithm in $\boldsymbol{V}$, which is as expected from Lemma 3. For bigger values of κ, however, we observe that the sensitivity-based algorithm works better. This can happen, because both Ipopt and the sensitivity-based algorithm find local optima and the sensitivity-based algorithm can find a better one.

We also consider the Sioux Falls network presented in Suwansirikul et al. (1987). Sioux Falls is a network with 24 nodes and 76 arcs and 576 OD pairs with $n = 4$. Due to the large size of the Sioux Falls network problems and the difficulty of the problems, the Ipopt solver takes more than 24 hours to solve the Sioux Falls network problems, unless it generates errors and fails to find a solution. We only present our results obtained by the sensitivity-based algorithm. Figure 3b represents the total travel time in the case of PRUE, and the worst-cases of MSatUE. Since we only consider $\boldsymbol{V}$ in the sensitivity-based algorithm, the actual bounds for the worst-cases can be larger, respectively.

Figure 4a compares the total travel time in the worst-case scenario with the analytical bound provided in Theorem 4 for MSatUE for the nine-node network; similarly Figure 4b for the Sioux



(a) The nine-node network of Hearn and Ramana (1998)

(b) The Sioux Falls network

Figure 4: Comparing with the analytical bound

Falls network. We observe that there is a large gap between the analytical and numerical bounds. Although the analytical result certainly provides a valid bound, it is too large to be practically useful in realistic road networks. This indicates opportunities for empirical studies on the bounds and other analytical bounds that depend on more network-specific information such as travel demands and travel time functions. The bound $(1+\kappa)^{n+1}$ in Theorem 4 is independent from such network-specific information.

5.3.2 Asymmetric Travel Time Function Networks

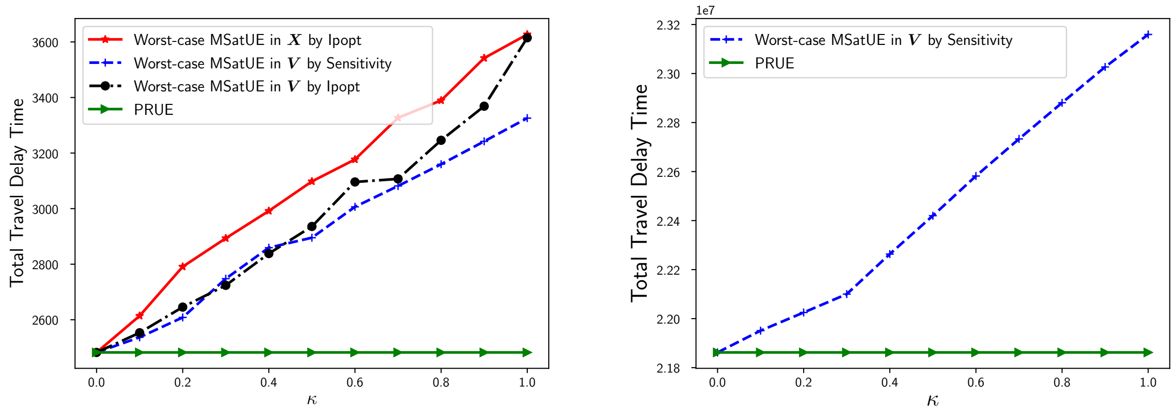
In this section, we compare the worst case total travel times of satisficing user equilibrium and the perfectly rational user equilibrium. We also compare the results obtained by the sensitivity based algorithm, Ipopt solver in $\boldsymbol{V}$, and Ipopt solver in $\boldsymbol{X}$.

We first create a asymmetric variant of the nine-node network as shown in Figure 6 in Appendix A. As presented in Figure 5a, the total travel time obtained by sensitivity-based algorithm and Ipopt solver in $\boldsymbol{V}$ are pretty closed specially for smaller values of κ. However, Ipopt solver in $\boldsymbol{X}$ outperforms both sensitivity-based algorithm and Ipopt solver in $\boldsymbol{V}$, as it is as expected from Lemma 3. We use the fixed-point method to solve (48) and (47) in the sensitivity based algorithm.

Figure 5b represents the total travel time in the case of PRUE, and the worst-cases of MSatUE for the Sioux Falls network considering asymmetric cost function (62) in Appendix A.

6 Concluding Remarks

When network users are satisficing decision makers, the resulting satisficing user equilibria may lose the system performance, compared to the perfectly rational user equilibrium. To quantify how much we can lose, this paper has quantified the worst-case analytical bound on the price of satisficing and



(a) The nine-node network

(b) The Sioux Falls network

Figure 5: Comparing total travel times with asymmetric travel time functions

suggested a numerical algorithm based on sensitivity analysis.

We suggest potential future research directions both in analytical and numerical bounds. For the analytical bound, our result is based on the condition (36). By attempting to relax this condition, one may obtain a global bound for any value of κ.

To improve the numerical bounds, we need to consider the problem in $\boldsymbol{X}$, instead of $\boldsymbol{V}$ as done in this paper. The critical challenge of applying the sensitivity analysis results in $\boldsymbol{X}$ is that solutions to the equilibrium problems in $\boldsymbol{X}$ are not unique in general; hence differentiability with respect to perturbations cannot be guaranteed. Algorithms without requiring differentiability are necessary to solve the problems in $\boldsymbol{X}$.

In deriving the analytical bounds, we utilized a novel technique comparing equilibrium patterns before and after the travel demand is increased; namely $\boldsymbol{V}$ and $\boldsymbol{V}_{1+\kappa}$. Applying this technique in the context of the burden of risk aversion and the deviation ratio would be an interesting research direction.

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Appendices

A Nine-node Asymmetric Networks

In order to test the performance of sensitivity-based algorithm in an asymmetric network, we create an asymmetric version of the nine-node network considered by Hearn and Ramana (1998). In the asymmetric nine-node network, which has been shown in figure 6, we add a few additional links and assume that the link travel cost function is:

$$t_a(\boldsymbol{v}) = A_a + B_a\left(\frac{0.5v_{\widehat{a}} + v_a}{C_a}\right)^4 \tag{62}$$

where $\widehat{a}$ is the flow in the opposite link. Thus, the link travel function depends not only on the flow in that link, but also on the flow in the link in opposite direction. The values of parameters A_a, B_a and C_a are given in Table 1 for each link.

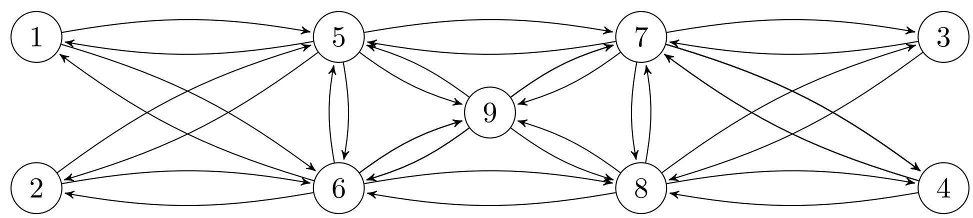

Figure 6: Asymmetric nine-node network

Table 1: asymmetric nine-node network link cost function parameters

a	A_a	B_a	C_a
(1,5)	12	1.80	5
(1,6)	18	2.70	6
(2,5)	35	5.25	3
(2,6)	35	5.25	9
(5,6)	20	3.00	9
(5,7)	11	1.65	2
(5,9)	26	3.90	8
(6,8)	33	4.95	6
(6,9)	30	4.50	8
(7,3)	25	3.75	3
(7,4)	24	3.60	6
(7,8)	19	2.85	2
(8,3)	39	5.85	8
(8,4)	43	6.45	6
(9,7)	26	3.90	4
(9,8)	30	4.50	8
(5,1)	12	1.80	5
(6,1)	18	2.70	6
(5,2)	35	5.25	3
(6,2)	35	5.25	9
(6,5)	20	3.00	9
(7,5)	11	1.65	2
(9,5)	26	3.90	8
(8,6)	33	4.95	6
(9,6)	30	4.50	8
(3,7)	25	3.75	3
(4,7)	24	3.60	6
(8,7)	19	2.85	2
(3,8)	39	5.85	8
(4,8)	43	6.45	6
(7,9)	26	3.90	4
(8,9)	30	4.50	8