

On the Price of Satisficing in Network User Equilibria

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Abstract

When network users are satisficing decision-makers, the resulting traffic pattern attains a satisficing user equilibrium, which may deviate from the (perfectly rational) user equilibrium. In a satisficing user equilibrium traffic pattern, the total system travel time can be worse than in the case of the PRUE. We show how bad the worst-case satisficing user equilibrium traffic pattern can be, compared to the perfectly rational user equilibrium. We call the ratio between the total system travel times of the two traffic patterns the price of satisficing, for which we provide an analytical bound. Using the sensitivity analysis for variational inequalities, we propose a numerical method to quantify the price of satisficing for any given network instance.

Keywords: bounded rationality; satisficing; user equilibrium; sensitivity analysis

1 Introduction

Instead of assuming a perfectly rational person with a clear system of preferences and perfect knowledge of the surrounding decision-making environment, we can consider *boundedly* rational persons with (1) an ambiguous system of preferences and (2) lack of complete information, following Simon (1955). When decision makers are indifferent among alternatives within a certain threshold, they are called *satisficing* decision makers, opposed to *optimizing* decision makers. The notion of satisficing was first introduced by Simon (1955, 1956). Satisficing decision makers choose any alternative whose utility level is above a threshold, called an *aspiration level*, even when the alternative is not optimal. The satisficing behavior is related to the first source of boundedness—an ambiguous system of preferences.

In transportation research, modeling drivers' route choice is an important task. While the travel-time minimization has been traditionally used as a basis for such modeling, sub-optimal route-choice behavior has gained attention. Since Mahmassani and Chang (1987), bounded rationality has gained attention in the transportation research literature (Szeto and Lo, 2006; Wu et al., 2013;

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Han et al., 2015; Szeto and Lo, 2006; Ge and Zhou, 2012; Di et al., 2014; Guo, 2013; Lou et al., 2010). Empirical evidence supports bounded rationality of drivers (Nakayama et al., 2001; Zhu and Levinson, 2010). The notion of bounded rationality has also been considered in the evaluation of value of times in connection to route-choice modeling (Xu et al., 2017), and in the model of behavior adjustment process (Ye and Yang, 2017). We refer readers to a review of Di and Liu (2016). In the non-transportation literature, the notion of bounded rationality and satisficing has also received much attention (Charnes and Cooper, 1963; Lam et al., 2013; Jaillet et al., 2016; Chen et al., 1997; Brown and Sim, 2009).

While the above-mentioned transportation research literature considers boundedly rational drivers, their discussion is limited to satisficing drivers without considering the second source of boundedness: lack of complete information on the decision environment. Sun et al. (2017) connect the first and the second sources of boundedness by considering both satisficing behavior and incomplete information, in the context of shortest-path finding in *congestion-free* networks. Sun et al. (2017) study the second source by considering errors in drivers' perception of arc travel time, and conclude that their perception-error model can generally capture both sources of boundedness in rationality in a single unified modeling framework.

In the literature, the traditional network user equilibrium, Wardrop equilibrium in particular, is called the perfectly rational user equilibrium (PRUE), while a traffic pattern equilibrated among satisficing drivers is called a boundedly rational user equilibrium (BRUE). In this paper, we will use a new term *satisficing user equilibrium (SatUE)* instead of BRUE to emphasize that it only considers the first source of boundedness without considering drivers' incomplete information on the decision environment.

The main contribution of this paper is the quantification of how bad the total system travel time in SatUE can be. In a SatUE traffic pattern, the total system travel time can be either greater than or less than that of PRUE. We define the *price of satisficing (PoSat)* as the ratio between the worst-case total system travel time of SatUE and the total system travel time of PRUE. This paper quantifies PoSat both theoretically and numerically.

The theoretical quantification of PoSat is related to the price of anarchy (PoA) (Koutsoupias and Papadimitriou, 1999; Roughgarden and Tardos, 2002) that compares the performances of the system optimal solutions and the PRUE solutions. Using a similar idea, we can also compare the performance of the perfectly rational user equilibrium traffic patterns and satisficing user equilibrium traffic patterns. While PoA quantifies how much system-wide performance we can lose by competing, PoSat quantifies how much we can lose by satisficing. Roughgarden and Tardos (2002) define and study the PoA of *approximate Nash equilibria*, which are essentially SatUE patterns. We develop the bound on PoSat by learning from the PoA of approximate Nash equilibria (Christodoulou et al., 2011) and incorporating the ideas from the sensitivity analysis of traffic equilibria (Dafermos and Nagurney, 1984) with a novel technique.

The notion of PoSat is also related to the *burden of risk aversion* (Nikolova and Stier-Moses, 2015) and the *deviation ratio* (Kleer and Schäfer, 2016). When network users are risk-averse decision

makers, the burden of risk aversion compares the performances of the resulting equilibrium among risk-averse users and the (risk-neutral) PRUE. When network users' cost functions are deviated from the true cost functions for some reasons, the deviation ratio compares the performances of the resulting equilibrium and the PRUE. Kler and Schäfer (2016) show that the burden of risk aversion is a special case of the deviation ratio. In both research articles, however, only cases with a common single origin node are considered. In this paper, we consider general cases with multiple origin nodes and multiple destination nodes.

The numerical quantification of PoSat utilizes the sensitivity analysis of parametric variational inequalities. We can present sufficient conditions for SatUE in the form of parametric variational inequalities, which we call the user equilibrium with perception errors (UE-PE). When a network instance is given, we can compute the worst-case performance of SatUE, by varying the perception parameters in the UE-PE model. This method requires information on derivatives of equilibrium solutions with respect to the perception parameters.

Tobin and Friesz (1988) first compute derivatives in the sensitivity analysis of traffic equilibria, using the implicit function theorem and applying ideas from sensitivity analysis for nonlinear programming problems. Friesz et al. (1990) apply the derivative result in solving mathematical programming with equilibrium constraints (MPEC). Later, using an affine variable transformation, Patriksson and Rockafellar (2003) and Patriksson (2004) provide derivatives under weaker regularity conditions for traffic equilibria, and Josefsson and Patriksson (2007) apply the method to network design problems. Chung et al. (2014) compare methods of sensitivity analysis and discuss the conditions under which the method of Tobin and Friesz (1988) is valid. We utilize the sensitivity-based method of Patriksson (2004) and Josefsson and Patriksson (2007).

This paper is organized as follows. In Section 2, we introduce the notation and define various concepts including user equilibrium, system optimum, satisficing behavior, price of anarchy, and price of satisficing. In Section 3, we define the user equilibrium with perception errors and make connections with satisficing user equilibrium. Our main result is introduced in Section 4, where we derive the theoretical worst-case bound on the price of satisficing. In Section 5, we show that a gradient projection method based on sensitivity analysis for variational inequalities can be applied to compute the best- and worst-case performances of satisficing user equilibrium, and demonstrate our results through numerical examples. Section 6 concludes this paper.

2 Notation and Definitions

Since we will use path-based and arc-based flow variables and their corresponding functions and sets interchangeably, we need clear definitions of variables, sets, and functions. We use boldfaced lower-case letters for vector quantities as in \mathbf{v} and normal lower-case letters for their components as in v_a ; similarly, vector-valued functions like $\mathbf{t}(\cdot)$ and their components like $t_a(\cdot)$. We use boldfaced upper-case letters for the set that they belong to, as in $\mathbf{v} \in \mathbf{V}$. We use calligraphic capital letters for sets of indices as in \mathcal{N} . The only exception is a vector Q with Q_w being its elements; we use q_i^w

for another value related to Q .

2.1 Traffic Flow Variables and Feasible Sets

We consider a network with a set of origin and destination \mathcal{W} that is represented by directed graph $G(\mathcal{N}, \mathcal{A})$, where \mathcal{N} is the set of nodes, and \mathcal{A} is the set of arcs. For each OD pair $w \in \mathcal{W}$, the travel demand is Q_w and the set of available paths is \mathcal{P}_w . The set of all available paths in the whole network is defined as $\mathcal{P} = \cup_{w \in \mathcal{W}} \mathcal{P}_w$.

We also define the set of path flow variables \mathbf{f} as

$$\mathbf{F} = \left\{ \mathbf{f} : \sum_{p \in \mathcal{P}_w} f_p = Q_w \quad \forall w \in \mathcal{W}, \quad f_p \geq 0 \quad \forall p \in \mathcal{P} \right\}$$

and the corresponding set of arc flow variables \mathbf{v} is defined as

$$\mathbf{V} = \left\{ \mathbf{v} : v_a = \sum_{p \in \mathcal{P}} \delta_a^p f_p \quad \forall a \in \mathcal{A}, \quad \mathbf{f} \in \mathbf{F} \right\}$$

where $\delta_a^p = 1$ if path p contains arc a and $\delta_a^p = 0$ otherwise. Let \mathcal{A}_i^+ and \mathcal{A}_i^- be the set of arcs whose tail node and head node are i , respectively. When we need to preserve OD information in arc flow variables, we use \mathbf{x} as follows:

$$\begin{aligned} \mathbf{X} &= \left\{ \mathbf{x} : x_a^w = \sum_{p \in \mathcal{P}_w} \delta_a^p f_p \quad \forall a \in \mathcal{A}, w \in \mathcal{W} \quad \mathbf{f} \in \mathbf{F} \right\} \\ &= \left\{ \mathbf{x} : \sum_{a \in \mathcal{A}_i^+} x_a^w - \sum_{a \in \mathcal{A}_i^-} x_a^w = q_i^w \quad \forall w \in \mathcal{W}, i \in \mathcal{N} \right\}. \end{aligned}$$

where $q_i^w = -Q_w$ if $i = o(w)$, $q_i^w = Q_w$ if $i = d(w)$, and $q_i^w = 0$ otherwise.

We have $v_a = \sum_{p \in \mathcal{P}} \delta_a^p f_p$, $x_a^w = \sum_{p \in \mathcal{P}_w} \delta_a^p f_p$, and $v_a = \sum_{w \in \mathcal{W}} x_a^w$. Therefore, the transformations from \mathbf{f} to \mathbf{v} , from \mathbf{f} to \mathbf{x} , and from \mathbf{x} to \mathbf{v} are unique, which are denoted by $\mathbf{f} \mapsto \mathbf{v}$, $\mathbf{f} \mapsto \mathbf{x}$, and $\mathbf{x} \mapsto \mathbf{v}$, respectively. The inverse transformations are, however, not unique. In the rest of this paper, to emphasize the non-uniqueness of the transformation and refer to *any* result of such transformation, we use $\xrightarrow{\text{any}}$; for example, with $\mathbf{v} \xrightarrow{\text{any}} \mathbf{f}$, we consider any \mathbf{f} such that $v_a = \sum_{p \in \mathcal{P}} \delta_a^p f_p$.

We will use \mathbf{v} , \mathbf{f} , and \mathbf{x} interchangeably to describe the same traffic pattern. In particular, we define

- \mathbf{f}^* , \mathbf{v}^* , \mathbf{x}^* : system optimal flow vectors (Section 2.2)
- \mathbf{f}^0 , \mathbf{v}^0 , \mathbf{x}^0 : perfectly rational user equilibrium flow vectors (Section 2.3)
- \mathbf{f}^κ , \mathbf{v}^κ , \mathbf{x}^κ : (multiplicative) satisficing user equilibrium flow vectors with κ (Section 2.4)

Note that when $\kappa = 0$, we have $\mathbf{f}^\kappa = \mathbf{f}^0$.

2.2 Travel Time Functions and System Optimum

We denote arc travel function with arc traffic volume v_a by $t_a(v_a)$. We consider a performance function for each arc a as

$$z_a(v_a) = t_a(v_a)v_a$$

which is assumed to be a convex function. We denote the travel time function along path p with flow \mathbf{f} by $c_p(\mathbf{f})$. When problems are stated with respect to \mathbf{x} , given $v_a = \sum_w x_a^w$, we define $\tau_a^w(\mathbf{x}) = \tau_a(\mathbf{x}) = t_a(v_a)$.

We can consider path-based performance function as follows

$$z_p(\mathbf{f}) = c_p(\mathbf{f})f_p$$

Arc travel function and flow travel function are related to each other.

$$c_p(\mathbf{f}) = \sum_{a \in \mathcal{A}} \delta_a^p t_a(v_a)$$

We define the arc-based total system performance function $Z(\mathbf{v})$ and path-based total system performance function $C(\mathbf{f})$ interchangeably as follows:

$$\begin{aligned} Z(\mathbf{v}) &\equiv \sum_{a \in \mathcal{A}} z_a(v_a) = \sum_{a \in \mathcal{A}} t_a(v_a)v_a \\ &= \sum_{p \in \mathcal{P}} z_p(\mathbf{f}) = \sum_{p \in \mathcal{P}} c_p(\mathbf{f})f_p = \sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}_w} c_p(\mathbf{f})f_p \equiv C(\mathbf{f}), \end{aligned}$$

which is also called the total system travel time. If a flow pattern minimizes $Z(\cdot)$ or $C(\cdot)$, it is called a *system optimal* flow pattern.

The vector valued function $\mathbf{t}(\cdot)$ is called *monotone* in \mathbf{V} if

$$[\mathbf{t}(\mathbf{v}^1) - \mathbf{t}(\mathbf{v}^2)]^\top (\mathbf{v}^1 - \mathbf{v}^2) \geq 0 \quad (1)$$

for all $\mathbf{v}^1, \mathbf{v}^2 \in \mathbf{V}$. If (1) holds as a strict inequality for all $\mathbf{v}^1 \neq \mathbf{v}^2$, it is said *strictly monotone*. The function $\mathbf{t}(\cdot)$ is called *strongly monotone* in \mathbf{V} with modulus $\alpha > 0$ if

$$[\mathbf{t}(\mathbf{v}^1) - \mathbf{t}(\mathbf{v}^2)]^\top (\mathbf{v}^1 - \mathbf{v}^2) \geq \alpha \|\mathbf{v}^1 - \mathbf{v}^2\|_{\mathbf{V}}^2 \quad (2)$$

for all $\mathbf{v}^1, \mathbf{v}^2 \in \mathbf{V}$, where $\|\cdot\|_{\mathbf{V}}$ is the l^2 -norm in \mathbf{V} . The monotonicity of path-based travel time function $c_p(\cdot)$ or its vector form $\mathbf{c}(\cdot)$ can be similarly defined.

2.3 Perfectly Rational User Equilibrium

When network users are perfectly rational, i.e. they seek the shortest path, we attain the perfectly rational user equilibrium (PRUE) defined as follows:

Definition 1 (Perfectly Rational User Equilibrium). A traffic pattern \mathbf{f}^0 is called a *perfectly rational user equilibrium* (PRUE), if

$$\text{(PRUE)} \quad f_p^0 > 0 \implies c_p(\mathbf{f}^0) = \min_{p' \in \mathcal{P}_w} c_{p'}(\mathbf{f}^0) \quad (3)$$

for all $p \in \mathcal{P}_w$ and $w \in \mathcal{W}$.

Using the arc travel function, the above condition can be restated as follows

$$f_p^0 > 0 \implies \sum_{a \in \mathcal{A}} \delta_a^p t_a(v_a^0) = \min_{p' \in \mathcal{P}_w} \sum_{a \in \mathcal{A}} \delta_a^{p'} t_a(v_a^0) \quad (4)$$

for all $p \in \mathcal{P}_w$ and $w \in \mathcal{W}$.

It is well known that a solution to the following variational inequality problem is a user equilibrium traffic flow (Smith, 1979; Dafermos, 1980):

$$\text{to find } \bar{\mathbf{f}} \in \mathbf{F} : \sum_{p \in \mathcal{P}} c_p(\bar{\mathbf{f}})(f_p - \bar{f}_p) \geq 0 \quad \forall \mathbf{f} \in \mathbf{F}, \quad (5)$$

which can be equivalently rewritten as:

$$\text{to find } \bar{\mathbf{v}} \in \mathbf{V} : \sum_{a \in \mathcal{A}} t_a(\bar{v}_a)(v_a - \bar{v}_a) \geq 0 \quad \forall \mathbf{v} \in \mathbf{V}, \quad (6)$$

or

$$\text{to find } \bar{\mathbf{x}} \in \mathbf{X} : \sum_{a \in \mathcal{A}} \sum_{w \in \mathcal{W}} \tau_a(\bar{\mathbf{x}})(x_a^w - \bar{x}_a^w) \geq 0 \quad \forall \mathbf{x} \in \mathbf{X} \quad (7)$$

with $\tau_a^w(\mathbf{x}) = \tau_a(\mathbf{x}) = t_a(v_a)$.

With strictly monotone functions $t_a(\cdot)$, the solution $\bar{\mathbf{v}}$ to (6) is unique. While the transformations $\bar{\mathbf{v}} \xrightarrow{\text{any}} \bar{\mathbf{f}}$ and $\bar{\mathbf{v}} \xrightarrow{\text{any}} \bar{\mathbf{x}}$ are not unique, any such $\bar{\mathbf{f}}$ and $\bar{\mathbf{x}}$ are solutions to (5) and (7), respectively; therefore, solutions to (5) and (7) are not unique in general.

2.4 Satisficing User Equilibrium

We introduce definitions of satisficing behavior and corresponding user equilibrium traffic patterns. A typical definition in the transportation research literature (e.g. Lou et al., 2010; Di et al., 2013; Han et al., 2015), termed boundedly rational user equilibrium (BRUE), uses an additive term. We call it additive satisficing user equilibrium, since bounded rationality could imply a broader concept than just satisficing behavior.

Definition 2 (Additive Satisficing). A traffic pattern \mathbf{f} is called an *additive satisficing user equilibrium* (ASatUE) with E , if

$$\text{(ASatUE)} \quad f_p > 0 \implies c_p(\mathbf{f}) \leq \min_{p' \in \mathcal{P}_w} c_{p'}(\mathbf{f}) + E \quad (8)$$

for all $p \in \mathcal{P}_w$ and $w \in \mathcal{W}$, where E is a positive constant.

A similar definition can be provided using a multiplicative term as follows:

Definition 3 (Multiplicative Satisficing). A traffic pattern \mathbf{f}^κ is called a *multiplicative satisficing user equilibrium* with κ , or κ -MSatUE, if

$$(\text{MSatUE}) \quad f_p^\kappa > 0 \implies c_p(\mathbf{f}^\kappa) \leq (1 + \kappa) \min_{p' \in \mathcal{P}_w} c_{p'}(\mathbf{f}^\kappa) \quad (9)$$

for all $p \in \mathcal{P}_w$ and $w \in \mathcal{W}$, where $\kappa \in [0, 1]$ is a constant.

Note that E in (8) and κ in (9) may be defined for each OD pair w , for example E_w and κ_w respectively, to allow non-homogeneous satisficing threshold for each OD pair w . In such cases, however, we assume that travelers for the same OD pair are homogeneous with the same threshold κ_w . In this paper, for simplicity, we use a single value of κ for all OD pairs.

2.5 Price of Satisficing

The price of anarchy (PoA) compares the performances of approximate Nash equilibrium and system optimum; $C(\mathbf{f}^\kappa)$ and $C(\mathbf{f}^*)$, respectively. Among possibly multiple approximate Nash equilibrium traffic patterns, we are interested in the worst-case. Let us denote the set of approximate Nash equilibrium for a given network instance ρ by $\Psi_\kappa(\rho)$. We can define PoA of a network instance ρ as follows:

$$\text{PoA}(\rho) = \max_{\mathbf{f}^\kappa \in \Psi_\kappa(\rho)} \frac{C(\mathbf{f}^\kappa)}{C(\mathbf{f}^*)}, \quad (10)$$

and we are usually interested in its upper bound among all network instances, $\sup_\rho \text{PoA}(\rho)$.

In the context of bounded rationality and satisficing, we are more interested in comparing the performances of approximate Nash equilibrium—equivalently MSatUE—and the perfectly rational user equilibrium; $C(\mathbf{f}^\kappa)$ and $C(\mathbf{f}^0)$, respectively. We define the price of satisficing (PoSat) of instance ρ as follows:

$$\text{PoSat}(\rho) = \max_{\mathbf{f}^\kappa \in \Psi_\kappa(\rho)} \frac{C(\mathbf{f}^\kappa)}{C(\mathbf{f}^0)}, \quad (11)$$

and its upper bound among all network instances, $\sup_\rho \text{PoSat}(\rho)$.

3 User Equilibrium with Perception Errors

Related to MSatUE, we introduce the user equilibrium with perception error (UE-PE) model. In this model, we assume that network users are optimizing, i.e. seeking the shortest path; however, we assume that users may have their own perception of the travel time function.

We let ε_a^w denote the perception error of travel time along arc a of users in OD pair w . A vector $\bar{\mathbf{x}} \in \mathbf{X}$ is a solution to the UE-PE model, if

$$\sum_{a \in \mathcal{A}} \sum_{w \in \mathcal{W}} (t_a(\bar{v}_a) - \varepsilon_a^w)(x_a^w - \bar{x}_a^w) \geq 0 \quad \forall \mathbf{x} \in \mathbf{X} \quad (12)$$

for *some* ε such that

$$0 \leq \varepsilon_a^w \leq \frac{\kappa}{1+\kappa} t_a(\bar{v}_a) \quad \forall a \in \mathcal{A}, w \in \mathcal{W}. \quad (13)$$

We note that $t_a(\bar{v}_a) - \varepsilon_a^w$ is the *perceived* travel time for drivers of OD pair w . The term ε_a^w represents the perception error for arc a and OD pair w . In this model, we assume all drivers for each OD pair are homogeneous in their perception of arc travel time.

With changes of variables $\lambda_a^w t_a(v_a) = t_a(v_a) - \varepsilon_a^w$, the UE-PE model (12) can be restated as follows:

$$\text{(UE-PE-}\mathbf{X}\text{)} \quad \sum_{a \in \mathcal{A}} \sum_{w \in \mathcal{W}} \lambda_a^w t_a(\bar{v}_a) (x_a^w - \bar{x}_a^w) \geq 0 \quad \forall \mathbf{x} \in \mathbf{X} \quad (14)$$

for some $\lambda_a^w \in [\frac{1}{1+\kappa}, 1]$ for all $w \in \mathcal{W}$ and $a \in \mathcal{A}$. We observe that the UE-PE model generates a subset of SMSatUE traffic flow patterns.

Lemma 1 (UE-PE- $\mathbf{X} \implies$ MSatUE). *Suppose $\bar{\mathbf{x}}$ is a solution to UE-PE- \mathbf{X} in (12) with some $\bar{\lambda}$ where $\bar{\lambda}_a^w \in [\frac{1}{1+\kappa}, 1]$ for all $w \in \mathcal{W}$ and $a \in \mathcal{A}$. Then any $\bar{\mathbf{f}}$ with $\bar{\mathbf{x}} \xrightarrow{\text{any}} \bar{\mathbf{f}}$ is a κ -MSatUE flow.*

Proof of Lemma 1. Given $\bar{\mathbf{f}}$, we let $\bar{\mathbf{v}}$ be the arc flow vector from $\bar{\mathbf{f}} \mapsto \bar{\mathbf{v}}$. Let $\bar{\varepsilon}$ is the perception error that makes $\bar{\mathbf{x}}$ a solution to (12). Under the perception error $\bar{\varepsilon}$, we know that $\bar{\mathbf{x}}$ is a user equilibrium traffic flow; hence the following condition holds from (4):

$$\bar{f}_p > 0 \implies \sum_{a \in \mathcal{A}} \delta_a^p \bar{\lambda}_a^w t_a(\bar{v}_a) = \min_{p' \in \mathcal{P}_w} \sum_{a \in \mathcal{A}} \delta_a^{p'} \bar{\lambda}_a^w t_a(\bar{v}_a) \quad (15)$$

for all $p \in \mathcal{P}_w$ and $w \in \mathcal{W}$. Since $\bar{\lambda}_a^w \in [\frac{1}{1+\kappa}, 1]$, the right-hand-side of (15) implies

$$\frac{1}{1+\kappa} \sum_{a \in \mathcal{A}} \delta_a^{p'} t_a(\bar{v}_a) \leq \min_{p' \in \mathcal{P}_w} \sum_{a \in \mathcal{A}} \delta_a^{p'} \bar{\lambda}_a^w t_a(\bar{v}_a) \leq \min_{p' \in \mathcal{P}_w} \sum_{a \in \mathcal{A}} \delta_a^{p'} t_a(\bar{v}_a),$$

which is equivalent to the following path flow form:

$$c_{p'}(\bar{\mathbf{f}}) \leq (1+\kappa) \min_{p' \in \mathcal{P}_w} \sum_{a \in \mathcal{A}} c_{p'}(\bar{\mathbf{f}}).$$

Therefore, we conclude that $\bar{\mathbf{f}}$ is a κ -MSatUE traffic flow. \square

We can also provide a path-based formulation of UE-PE:

$$\text{(UE-PE-}\mathbf{F}\text{)} \quad \sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}_w} \tilde{c}_p^w(\bar{\mathbf{f}}) (f_p - \bar{f}_p) \geq 0 \quad \forall \mathbf{f} \in \mathbf{F} \quad (16)$$

for the perceived path travel time functions $\tilde{c}_p^w(\mathbf{f}) = \sum_{a \in \mathcal{A}} \delta_a^p \lambda_a^w t_a(v_a)$ with *some* $\lambda_a^w \in [\frac{1}{1+\kappa}, 1]$.

Lemma 2 (UE-PE- $\mathbf{F} \iff$ UE-PE- \mathbf{X}). *If $\bar{\mathbf{f}} \in \mathbf{F}$ is a solution to UE-PE- \mathbf{F} in (16), then $\bar{\mathbf{x}}$ with $\bar{\mathbf{f}} \mapsto \bar{\mathbf{x}}$ is a solution to UE-PE- \mathbf{X} in (14). Conversely, if $\bar{\mathbf{x}} \in \mathbf{X}$ is a solution to UE-PE- \mathbf{X} in (14), then any $\bar{\mathbf{f}}$ with $\bar{\mathbf{x}} \xrightarrow{\text{any}} \bar{\mathbf{f}}$ is a solution to UE-PE- \mathbf{F} in (16).*

Proof of Lemma 2. We can prove both directions by observing that

$$\begin{aligned}
\sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}_w} \tilde{c}_p^w(\bar{\mathbf{f}})(f_p - \bar{f}_p) &= \sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}_w} \sum_{a \in \mathcal{A}} \delta_a^p \lambda_a^w t_a(\bar{v}_a)(f_p - \bar{f}_p) \\
&= \sum_{w \in \mathcal{W}} \sum_{a \in \mathcal{A}} \lambda_a^w t_a(\bar{v}_a) \left(\sum_{p \in \mathcal{P}_w} \delta_a^p f_p - \sum_{p \in \mathcal{P}_w} \delta_a^p \bar{f}_p \right) \\
&= \sum_{w \in \mathcal{W}} \sum_{a \in \mathcal{A}} \lambda_a^w t_a(\bar{v}_a)(x_a^w - \bar{x}_a^w).
\end{aligned}$$

□

When the values of λ_a^w are the same across all $w \in \mathcal{W}$, i.e. $\lambda_a = \lambda_a^w$ for all $w \in \mathcal{W}$, we can simplify (14) as follows:

$$(\text{UE-PE-V}) \quad \sum_{a \in \mathcal{A}} \lambda_a t_a(\bar{v}_a)(v_a - \bar{v}_a) \geq 0 \quad \forall \mathbf{v} \in \mathbf{V} \quad (17)$$

for *some* $\lambda_a \in [\frac{1}{1+\kappa}, 1]$ for each $a \in \mathcal{A}$. The simplified model (17) has been considered in the literature for approximate Nash equilibrium (Christodoulou et al., 2011) and Nash equilibrium with deviated travel time functions (Kleer and Schäfer, 2016). For the simplified model, we can state:

Lemma 3 (UE-PE-V \implies UE-PE-X). *Suppose that $\bar{\mathbf{v}} \in \mathbf{V}$ is a solution to UE-PE-V in (17). Let $\bar{\mathbf{x}}$ be any vector with $\bar{\mathbf{v}} \xrightarrow{\text{any}} \bar{\mathbf{x}}$. Then $\bar{\mathbf{x}}$ is a solution to UE-PE-X in (14).*

While Lemmas 1, 2, and 3 provide sufficient conditions for a traffic flow pattern to be a κ -MSatUE, Theorem 1 of Christodoulou et al. (2011) provides a necessary condition.

Lemma 4 (A necessary condition of MSatUE). *Let $\mathbf{f}^\kappa \in \mathbf{F}$ be a κ -MSatUE and $\mathbf{v}^\kappa \in \mathbf{V}$ be the corresponding arc flow vector with $\mathbf{f}^\kappa \mapsto \mathbf{v}^\kappa$. Then we have*

$$\sum_{a \in \mathcal{A}} t_a(v_a^\kappa)((1 + \kappa)v_a - v_a^\kappa) \geq 0 \quad \forall \mathbf{v} \in \mathbf{V}. \quad (18)$$

Christodoulou et al. (2011) derive a tight bound on the price of anarchy on approximate Nash equilibria based on Lemma 4. We conclude this section:

Theorem 1. *Implications in Lemmas 1–4 are summarized as follows:*

$$\begin{array}{c}
\text{UE-PE-V} \implies \text{UE-PE-X} \implies \text{MSatUE} \implies (18), \\
\updownarrow \\
\text{UE-PE-F}
\end{array}$$

where $X \implies Y$ means that any solution to X yields a solution to Y .

4 Bounding the Price of Satisficing

We first provide analytical bounds of $C(\mathbf{f}^\kappa)$ compared to $C(\mathbf{f}^0)$.

4.1 Lessons from the Price of Anarchy

We first observe that $\text{PoSat}(\rho) \leq \text{PoA}(\rho)$ for all network instance ρ , since $C(\mathbf{f}^0) \geq C(\mathbf{f}^*)$. This enables us to use the results from the price of anarchy literature for bounding PoSat. Theorem 2 of Christodoulou et al. (2011) provides the price of anarchy for the general polynomial cases, which immediately leads to the following result:

Lemma 5. *Suppose \mathbf{f}^κ is a κ -MSatUE flow, and $t_a(\cdot)$ is a polynomial with degree n . Define*

$$\zeta(\kappa, n) = \begin{cases} (1 + \kappa)^{(n+1)} & \text{if } \kappa \geq (n + 1)^{1/n} - 1, \\ \left(\frac{1}{1 + \kappa} - \frac{n}{(n+1)^{(n+1)/n}} \right)^{-1} & \text{if } 0 \leq \kappa \leq (n + 1)^{1/n} - 1. \end{cases} \quad (19)$$

Then we have

$$C(\mathbf{f}^*) \leq C(\mathbf{f}^\kappa) \leq \zeta(\kappa, n)C(\mathbf{f}^*) \leq \zeta(\kappa, n)C(\mathbf{f}^0). \quad (20)$$

That is, the PoSat is bounded above by $\zeta(\kappa, n)$.

Proof of Lemma 5. From Theorem 2 of Christodoulou et al. (2011), we have

$$C(\mathbf{f}^\kappa) \leq \zeta(\kappa, n)C(\mathbf{f}) \quad \forall \mathbf{f} \in \mathbf{F}. \quad (21)$$

Picking $\mathbf{f} = \mathbf{f}^0$ in (21), we obtain the upper bound on $C(\mathbf{f}^\kappa)$. Inequalities involving $C(\mathbf{f}^*)$ are from the fact $C(\mathbf{f}^*) \leq C(\mathbf{f})$ for all $\mathbf{f} \in \mathbf{F}$. \square

With the bounds of $C(\mathbf{f}^\kappa)$ given in Lemma 5, the real question is if $\zeta(\kappa, n)$ is a tight bound. Since $\zeta(\kappa, n)$ is obtained from the comparison between $C(\mathbf{f}^\kappa)$ and $C(\mathbf{f}^*)$, it is unclear if there is a case when we indeed have $C(\mathbf{f}^\kappa) = \zeta(\kappa, n)C(\mathbf{f}^0)$. We know for sure that $\zeta(\kappa, n)$ is not tight when κ is small, since $\zeta(0, n)$ is not equal to 1. We provide a partial answer for this question.

In Lemma 3 of Christodoulou et al. (2011), the existence of a network instance with $C(\mathbf{f}^\kappa) = (1 + \kappa)^{n+1}C(\mathbf{f}^*)$ is shown for $\kappa \geq (n + 1)^{1/n} - 1$ via an example. The same example is, however, valid for all $\kappa \geq 0$. Note that, in the same example, the system optimal flow is also at user equilibrium. Therefore, the worst-case PoSat is at least $(1 + \kappa)^{n+1}$ for all $\kappa \geq 0$. Thus, we obtain the following theorem:

Theorem 2. *When $0 \leq \kappa \leq (n + 1)^{1/n} - 1$, the worst-case PoSat falls in the following interval:*

$$(1 + \kappa)^{n+1} \leq \sup_{\rho} \text{PoSat}(\rho) \leq \left(\frac{1}{1 + \kappa} - \frac{n}{(n + 1)^{(n+1)/n}} \right)^{-1}. \quad (22)$$

When $\kappa \geq (n + 1)^{1/n} - 1$, the worst-case PoSat is exactly:

$$\sup_{\rho} \text{PoSat}(\rho) = (1 + \kappa)^{n+1}. \quad (23)$$

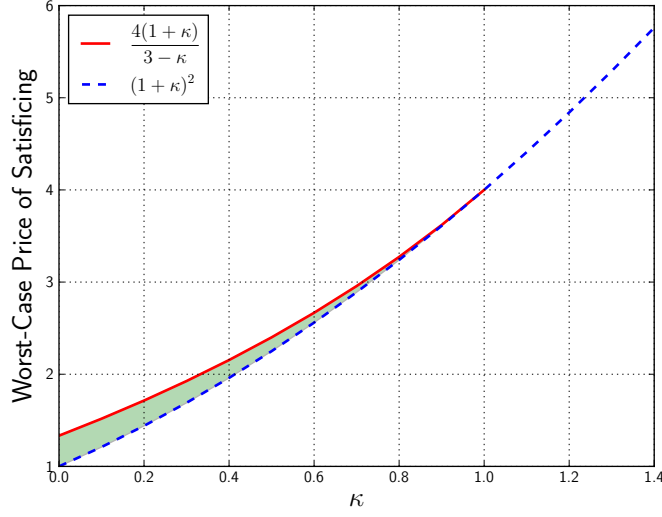


Figure 1: The worst-case price of satisficing with linear travel time functions. Note that when $n = 1$, the right-hand-side of (22) becomes $\frac{4(1+\kappa)}{3-\kappa}$. When $\kappa \geq 1$, we know for sure that the worst-case price of satisficing is exactly the dotted line. When $\kappa \leq 1$, the worst case falls in the shaded interval between the solid line and dotted line.

Figure 1 shows the bounds in Theorem 2 for the linear travel time function case. For smaller κ values, the worst-case PoSat falls in the shaded interval, while for larger κ values, it is exactly $(1 + \kappa)^2$. When κ is zero, we have $\mathbf{f}^\kappa = \mathbf{f}^0$; hence, we must have the PoSat approach to 1. With this observation, we naturally ask a question: Does $(1 + \kappa)^{n+1}$ provide a tight bound on the worst-case PoSat for all values of $\kappa \geq 0$? We present a partial answer to this question for a special case in the following section.

4.2 Tightening the Price of Satisficing

When the travel time functions $t_a(\cdot)$ are polynomials of order n , we show that $(1 + \kappa)^{n+1}$ provides the worst-case bound on the price of satisficing for small κ values, under an additional condition. We first define new sets of flow vectors. When the travel demand Q_w for each $w \in \mathcal{W}$ is multiplied by the factor $1 + \kappa$, we define

$$\begin{aligned} \mathbf{F}_{1+\kappa} &= \left\{ \mathbf{f} : \sum_{p \in \mathcal{P}_w} f_p = (1 + \kappa)Q_w \quad \forall w \in \mathcal{W}, \quad f_p \geq 0 \quad \forall p \in \mathcal{P} \right\}, \\ \mathbf{V}_{1+\kappa} &= \left\{ \mathbf{v} : v_a = \sum_{p \in \mathcal{P}} \delta_a^p f_p \quad \forall a \in \mathcal{A}, \quad \mathbf{f} \in \mathbf{F}_{1+\kappa} \right\}, \\ \mathbf{X}_{1+\kappa} &= \left\{ \mathbf{x} : x_a^w = \sum_{p \in \mathcal{P}_w} \delta_a^p f_p \quad \forall a \in \mathcal{A}, w \in \mathcal{W} \quad \mathbf{f} \in \mathbf{F}_{1+\kappa} \right\}. \end{aligned}$$

The above three sets can equivalently be written as follows:

$$\begin{aligned}\mathbf{F}_{1+\kappa} &= \{(1+\kappa)\mathbf{f} : \mathbf{f} \in \mathbf{F}\}, \\ \mathbf{V}_{1+\kappa} &= \{(1+\kappa)\mathbf{v} : \mathbf{v} \in \mathbf{V}\}, \\ \mathbf{X}_{1+\kappa} &= \{(1+\kappa)\mathbf{x} : \mathbf{x} \in \mathbf{X}\}.\end{aligned}$$

We will use ‘hat’ for flow vectors in these sets, for example, $\hat{\mathbf{f}}^\kappa \in \mathbf{F}_{1+\kappa}$, while without hat in the original sets as in $\mathbf{f}^\kappa \in \mathbf{F}$. For each $a \in \mathcal{A}$, we let

$$t_a(v_a) = \sum_{m=0}^n b_{am}(v_a)^m = b_{a0} + b_{a1}v_a + b_{a2}(v_a)^2 + \cdots + b_{an}(v_a)^n$$

for some constants b_{am} for $m = 0, 1, \dots, n$.

We characterize the multiple of flow vectors, $(1+\kappa)\mathbf{f}$, and their total system travel time, $C((1+\kappa)\mathbf{f})$. We first observe that the multiple of a PRUE flow, $(1+\kappa)\mathbf{f}^0$, provides a satisficing solution to the traffic equilibrium problem with the increased travel demand.

Lemma 6. *Suppose $t_a(\cdot)$ are polynomials of order n . If $\mathbf{f}^0 \in \mathbf{F}$ is a PRUE flow, then $(1+\kappa)\mathbf{f}^0$ is a σ -MSatUE flow with $\sigma = (1+\kappa)^n - 1$ in $\mathbf{F}_{1+\kappa}$. When $n = 1$, we have $\sigma = \kappa$.*

Proof. Let $\bar{\mathbf{f}} = (1+\kappa)\mathbf{f}^0$, and $\bar{\mathbf{v}} = (1+\kappa)\mathbf{v}^0$ for the corresponding arc flow vectors. If the condition

$$\sum_{a \in \mathcal{A}} \left(\sum_{m=0}^n \lambda_{am} b_{am}(\bar{v}_a)^m \right) (v'_a - \bar{v}_a) \geq 0 \quad \forall \mathbf{v}' \in \mathbf{V}_{1+\kappa} \quad (24)$$

holds for some constants $\lambda_{am} \in [\frac{1}{1+\sigma}, 1]$ for $m = 0, 1, \dots, n$ and $a \in \mathcal{A}$, then we can find $\lambda_a \in [\frac{1}{1+\sigma}, 1]$ such that

$$\lambda_a \sum_{m=0}^n b_{am}(\bar{v}_a)^m = \sum_{m=0}^n \lambda_{am} b_{am}(\bar{v}_a)^m$$

for all $a \in \mathcal{A}$. Therefore, by Theorem 1, $\bar{\mathbf{f}}$ is a σ -MSatUE flow in $\mathbf{F}_{1+\kappa}$.

Since \mathbf{v}^0 is PRUE for \mathbf{V} , we know that

$$\sum_{a \in \mathcal{A}} \left(\sum_{m=0}^n b_{am}(v_a^0)^m \right) (v_a - v_a^0) \geq 0 \quad \forall \mathbf{v} \in \mathbf{V}.$$

Therefore

$$\sum_{a \in \mathcal{A}} \left(\sum_{m=0}^n \frac{1}{(1+\kappa)^m} b_{am}((1+\kappa)v_a^0)^m \right) ((1+\kappa)v_a - (1+\kappa)v_a^0) \geq 0 \quad \forall \mathbf{v} \in \mathbf{V}$$

Letting for all $a \in \mathcal{A}$

$$\lambda_{am} = \frac{1}{(1+\kappa)^m}, \quad m = 0, 1, \dots, n$$

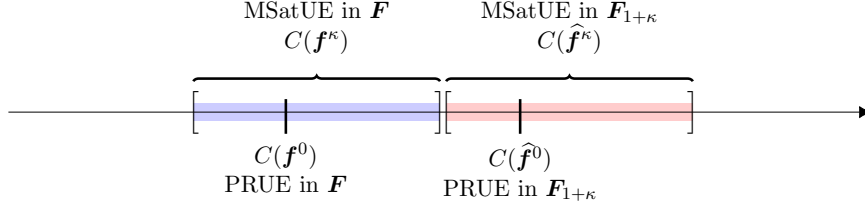


Figure 2: We know that $(1 + \kappa)C(\mathbf{f}^0) \leq C(\widehat{\mathbf{f}}^0)$ from Lemma 8. Is $C(\mathbf{f}^\kappa) \leq C(\widehat{\mathbf{f}}^\kappa)$ always? In other words, do the two shaded intervals have overlaps or not? We show that there is no overlap, except at the boundary, under a special condition in Lemma 7.

$$\begin{aligned}\bar{v}_a &= (1 + \kappa)v_a^0, \\ v'_a &= (1 + \kappa)v_a,\end{aligned}$$

we observe that $\lambda_{am} \in [\frac{1}{1+\sigma}, 1]$ and we obtain (24); hence proof. \square

Comparing the PRUE flows $\mathbf{f}^0 \in \mathbf{F}$ and $\widehat{\mathbf{f}}^0 \in \mathbf{F}_{1+\kappa}$, we can observe

$$(1 + \kappa)C(\mathbf{f}^0) \leq C(\widehat{\mathbf{f}}^0),$$

as shown in Appendix A. Can we obtain a similar result for comparing MSatUE flows? Figure 2 demonstrates these comparisons. By introducing an additional condition, we compare MSatUE flows with the proportional travel demand increase.

Lemma 7. *Let $\mathbf{f}^\kappa \in \mathbf{F}$ be any κ -MSatUE and $\widehat{\mathbf{f}}^\sigma \in \mathbf{F}_{1+\kappa}$ be any σ -MSatUE flows with the corresponding travel demands, when $\sigma = (1 + \kappa)^n - 1$. Suppose that $\kappa \geq 0$ is sufficiently small, in particular, so that*

$$\sum_{p \in \mathcal{P}} [c_p(\widehat{\mathbf{f}}^\sigma) - c_p(\mathbf{f}^\kappa)] (\widehat{f}_p^\sigma - f_p^\kappa) \geq \sigma \sum_{p \in \mathcal{P}} \max\{c_p(\widehat{\mathbf{f}}^\sigma), c_p(\mathbf{f}^\kappa)\} |\widehat{f}_p^\sigma - f_p^\kappa|. \quad (25)$$

Then we have

$$C(\mathbf{f}^\kappa) \leq C(\widehat{\mathbf{f}}^\sigma). \quad (26)$$

Proof of Lemma 7. We decompose \mathcal{P}_w for each OD pair w into the following four subsets:

$$\begin{aligned}\mathcal{P}_w^1 &= \{p \in \mathcal{P}_w : \widehat{f}_p^\sigma > 0, f_p^\kappa > 0, \widehat{f}_p^\sigma - f_p^\kappa \geq 0\}, \\ \mathcal{P}_w^2 &= \{p \in \mathcal{P}_w : \widehat{f}_p^\sigma > 0, f_p^\kappa > 0, \widehat{f}_p^\sigma - f_p^\kappa < 0\}, \\ \mathcal{P}_w^3 &= \{p \in \mathcal{P}_w : \widehat{f}_p^\sigma > 0, f_p^\kappa = 0\}, \\ \mathcal{P}_w^4 &= \{p \in \mathcal{P}_w : \widehat{f}_p^\sigma = 0, f_p^\kappa > 0\}.\end{aligned}$$

We ignore cases with $\widehat{f}_p^\sigma = 0$ and $f_p^\kappa = 0$. Note that $\widehat{f}_p^\sigma - f_p^\kappa > 0$ for $p \in \mathcal{P}_w^3$ and $\widehat{f}_p^\sigma - f_p^\kappa < 0$ for

$p \in \mathcal{P}_w^4$. From the definition of MSatUE flows, we have

$$\begin{aligned}\widehat{f}_p^\sigma > 0 &\implies c_p(\widehat{\mathbf{f}}^\sigma) \leq (1 + \sigma)\mu_w(\widehat{\mathbf{f}}^\sigma), \\ f_p^\kappa > 0 &\implies c_p(\mathbf{f}^\kappa) \leq (1 + \kappa)\mu_w(\mathbf{f}^\kappa),\end{aligned}$$

for all $p \in \mathcal{P}_w, w \in \mathcal{W}$. In addition, $\mu_w(\widehat{\mathbf{f}}^\sigma) \leq c_p(\widehat{\mathbf{f}}^\sigma)$ and $\mu_w(\mathbf{f}^\kappa) \leq c_p(\mathbf{f}^\kappa)$ for all $p \in \mathcal{P}$ by definition. Therefore, we have

$$\begin{aligned}& \sum_{p \in \mathcal{P}} [c_p(\widehat{\mathbf{f}}^\sigma) - c_p(\mathbf{f}^\kappa)](\widehat{f}_p^\sigma - f_p^\kappa) \\ & \leq \sum_{w \in \mathcal{W}} \left\{ \sum_{p \in \mathcal{P}_w^1} [(1 + \sigma)\mu_w(\widehat{\mathbf{f}}^\sigma) - \mu_w(\mathbf{f}^\kappa)](\widehat{f}_p^\sigma - f_p^\kappa) + \sum_{p \in \mathcal{P}_w^2} [\mu_w(\widehat{\mathbf{f}}^\sigma) - (1 + \kappa)\mu_w(\mathbf{f}^\kappa)](\widehat{f}_p^\sigma - f_p^\kappa) \right. \\ & \quad \left. + \sum_{p \in \mathcal{P}_w^3} [(1 + \sigma)\mu_w(\widehat{\mathbf{f}}^\sigma) - \mu_w(\mathbf{f}^\kappa)](\widehat{f}_p^\sigma - f_p^\kappa) + \sum_{p \in \mathcal{P}_w^4} [\mu_w(\widehat{\mathbf{f}}^\sigma) - (1 + \kappa)\mu_w(\mathbf{f}^\kappa)](\widehat{f}_p^\sigma - f_p^\kappa) \right\} \\ & = \sum_{w \in \mathcal{W}} \left\{ \sum_{p \in \mathcal{P}_w} [\mu_w(\widehat{\mathbf{f}}^\sigma) - \mu_w(\mathbf{f}^\kappa)](\widehat{f}_p^\sigma - f_p^\kappa) + \sigma \sum_{p \in \mathcal{P}_w^1 \cup \mathcal{P}_w^3} \mu_w(\widehat{\mathbf{f}}^\sigma)(\widehat{f}_p^\sigma - f_p^\kappa) \right. \\ & \quad \left. - \kappa \sum_{p \in \mathcal{P}_w^2 \cup \mathcal{P}_w^4} \mu_w(\mathbf{f}^\kappa)(\widehat{f}_p^\sigma - f_p^\kappa) \right\} \\ & \leq \sum_{w \in \mathcal{W}} \left\{ \sum_{p \in \mathcal{P}_w} [\mu_w(\widehat{\mathbf{f}}^\sigma) - \mu_w(\mathbf{f}^\kappa)](\widehat{f}_p^\sigma - f_p^\kappa) + \sigma \sum_{p \in \mathcal{P}_w} \max\{\mu_w(\widehat{\mathbf{f}}^\sigma), \mu_w(\mathbf{f}^\kappa)\} |\widehat{f}_p^\sigma - f_p^\kappa| \right\} \\ & \leq \sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}_w} [\mu_w(\widehat{\mathbf{f}}^\sigma) - \mu_w(\mathbf{f}^\kappa)](\widehat{f}_p^\sigma - f_p^\kappa) + \sigma \sum_{p \in \mathcal{P}} \max\{c_p(\widehat{\mathbf{f}}^\sigma), c_p(\mathbf{f}^\kappa)\} |\widehat{f}_p^\sigma - f_p^\kappa|.\end{aligned}$$

From (25), we obtain

$$\begin{aligned}0 &\leq \sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}} [\mu_w(\widehat{\mathbf{f}}^\sigma) - \mu_w(\mathbf{f}^\kappa)](\widehat{f}_p^\sigma - f_p^\kappa) \\ &= \sum_{w \in \mathcal{W}} [\mu_w(\widehat{\mathbf{f}}^\sigma) - \mu_w(\mathbf{f}^\kappa)] \left(\sum_{p \in \mathcal{P}} \widehat{f}_p^\sigma - \sum_{p \in \mathcal{P}} f_p^\kappa \right) \\ &= \sum_{w \in \mathcal{W}} [\mu_w(\widehat{\mathbf{f}}^\sigma) - \mu_w(\mathbf{f}^\kappa)] (\widehat{Q}_w - Q_w) \\ &= \kappa \sum_{w \in \mathcal{W}} \mu_w(\widehat{\mathbf{f}}^\sigma) Q_w - \kappa \sum_{w \in \mathcal{W}} \mu_w(\mathbf{f}^\kappa) Q_w \\ &= \frac{\kappa}{1 + \kappa} \sum_{w \in \mathcal{W}} \mu_w(\widehat{\mathbf{f}}^\sigma) \widehat{Q}_w - \kappa \sum_{w \in \mathcal{W}} \mu_w(\mathbf{f}^\kappa) Q_w \\ &\leq \frac{\kappa}{1 + \kappa} \sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}_w} c_p(\widehat{\mathbf{f}}^\sigma) \widehat{f}_p^\sigma - \frac{\kappa}{1 + \kappa} \sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}_w} c_p(\mathbf{f}^\kappa) f_p^\kappa \\ &= \frac{\kappa}{1 + \kappa} C(\widehat{\mathbf{f}}^\sigma) - \frac{\kappa}{1 + \kappa} C(\mathbf{f}^\kappa),\end{aligned}$$

which completes the proof. \square

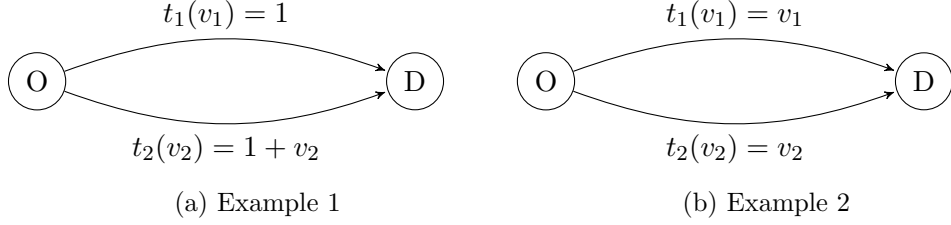


Figure 3: Examples where the travel demand is Q from node O to node D.

Lemma 7 leads to the following result:

Theorem 3. *Under the conditions in Lemma 7, we have*

$$C(\mathbf{f}^\kappa) \leq (1 + \kappa)^{n+1} C(\mathbf{f}^0), \quad (27)$$

hence $\sup_\rho \text{PoSat}(\rho) = (1 + \kappa)^{n+1}$.

Proof of Theorem 3. From Lemmas 6 and 7, we have

$$\begin{aligned} C(\mathbf{f}^\kappa) &\leq C((1 + \kappa)\mathbf{f}^0) = Z((1 + \kappa)\mathbf{v}^0) = \sum_{a \in \mathcal{A}} \left(\sum_{m=0}^n b_{am} ((1 + \kappa)v_a^0)^m \right) (1 + \kappa)v_a^0 \\ &\leq (1 + \kappa)^{n+1} \left(\sum_{m=0}^n b_{am} (v_a^0)^m \right) v_a^0 \\ &= (1 + \kappa)^{n+1} Z(\mathbf{v}^0) \\ &= (1 + \kappa)^{n+1} C(\mathbf{f}^0) \end{aligned}$$

where \mathbf{v}^0 is the arc flow vector from $\mathbf{f}^0 \mapsto \mathbf{v}^0$. □

For the illustration purpose, we consider two examples in Figure 3 with linear travel time functions, where $n = 1$ and $\sigma = \kappa$. In Example 1, the travel time function in the first arc is not increasing. We can verify that

$$\max C(\mathbf{f}^\kappa) = \begin{cases} Q + \kappa^2 & \text{if } \kappa \leq Q, & \text{with } \mathbf{f}^\kappa = (Q - \kappa, \kappa) \\ (1 + Q)Q & \text{if } \kappa \geq Q, & \text{with } \mathbf{f}^\kappa = (0, Q) \end{cases}$$

among all κ -MSatUE flows in \mathbf{F} and

$$\min C(\widehat{\mathbf{f}}^\kappa) = (1 + \kappa)Q \quad \text{with } \widehat{\mathbf{f}}^\kappa = (1 + \kappa)\mathbf{f}^0 = ((1 + \kappa)Q, 0).$$

among all κ -MSatUE flows in $\mathbf{F}_{1+\kappa}$. Comparing the two quantities, we observe $C(\mathbf{f}^\kappa) \leq C(\widehat{\mathbf{f}}^\kappa)$ in both cases. To show (26) in Lemma 7, condition (25) needs to hold only for these two flow vectors. Regardless of the value of κ , however, it is impossible to satisfy condition (25), although (26) still holds for all $\kappa \geq 0$. The price of satisficing is $1 + \frac{\kappa^2}{Q}$ if $\kappa < Q$ and $1 + Q$ if $\kappa \geq Q$ in this example,

both of which are less than $(1 + \kappa)^2$.

On the other hand, in Example 2, we have strictly monotone travel time functions in both arcs. Similarly, we consider

$$\begin{aligned} \max C(\mathbf{v}^\kappa) &= \frac{2 + 2\kappa + \kappa^2}{(2 + \kappa)^2} Q & \text{with } \mathbf{f}^\kappa &= \left(\frac{Q}{2 + \kappa}, \frac{(1 + \kappa)Q}{2 + \kappa} \right) \\ \min C(\widehat{\mathbf{v}}^\kappa) &= \frac{(1 + \kappa)^2}{2} Q & \text{with } \widehat{\mathbf{f}}^\kappa &= (1 + \kappa)\mathbf{f}^0 = \left(\frac{(1 + \kappa)Q}{2}, \frac{(1 + \kappa)Q}{2} \right) \end{aligned}$$

and can verify that $C(\mathbf{f}^\kappa) \leq C(\widehat{\mathbf{f}}^\kappa)$ for all $\kappa \geq 0$. In Example 2, we note that (25) holds for $\kappa \leq 0.206$. In this example, we observe that the price of satisficing is $\frac{2(2+2\kappa+\kappa^2)}{(2+\kappa)^2}$, which is no greater than $(1 + \kappa)^2$ for all $\kappa \geq 0$.

We have shown that Lemma 7 and Theorem 3 hold under condition (25). It still remains an open question if the results hold for general cases for all $\kappa \geq 0$. In the next sections, we will review other possible approaches and present numerical methods for computing the price of satisficing.

4.3 Other Approaches

When there is a single origin and multiple destinations, i.e., a single common origin node, in the network, Kler and Schäfer (2016) introduces the notion of the *deviation ratio* that compares the system performances of the user equilibrium and the equilibrium with *deviated* travel time functions $\tilde{t}_a(\cdot)$. The notion deviation may also be interpreted as perception in our definition. In a special case, the deviation ratio is reduced to the price of risk aversion (Nikolova and Stier-Moses, 2015) that compares the performances of equilibria among risk-averse and risk-neutral network users.

Kler and Schäfer (2016) define the deviated travel time functions with the following bounds:

$$t_a(v_a) + \alpha t_a(v_a) \leq \tilde{t}_a(v_a) \leq t_a(v_a) + \beta t_a(v_a) \quad (28)$$

where $-1 \leq \alpha \leq 0 \leq \beta$. The consideration of this deviated travel time function generalizes our UE-PE model where $\alpha = -\frac{\kappa}{1+\kappa}$ and $\beta = 0$. Kler and Schäfer (2016) show that the worst-case deviation ratio with (28) is bounded by

$$1 + \frac{\beta - \alpha}{1 + \alpha} \left\lceil \frac{|\mathcal{N}| - 1}{2} \right\rceil Q.$$

Therefore, we obtain the following theorem:

Theorem 4 (Kler and Schäfer, 2016). *Consider a directed graph with a single common origin node with the total travel demand Q and let $|\mathcal{N}|$ be the number of nodes. Then we have*

$$\frac{Z(\mathbf{v}^\kappa)}{Z(\mathbf{v}^0)} \leq 1 + \kappa \left\lceil \frac{|\mathcal{N}| - 1}{2} \right\rceil Q \quad (29)$$

where \mathbf{v}^κ is a solution to UE-PE- \mathbf{V} in (17).

Note that Theorem 4 only covers a subset of the entire MSatUE flows, as it is limited to the solutions UE-PE- \mathbf{V} in (17) and is applicable to cases with a *single* common origin. When Theorem 4 is applied in the examples in Figure 3, the bound (29) becomes $1 + \kappa Q$.

In another approach, the UE-PE model (12) can be interpreted as a parametric variational inequality that depends on the parameter ε . When the satisficing users' behavior is described by (12), we can provide bounds on the performance of the resulting traffic pattern, extending the sensitivity analysis result of Dafermos and Nagurney (1984) on parametric variational inequalities.

Given $v_a = \sum_w x_a^w$, we define $\tau_a^w(\mathbf{x}) = t_a(v_a)$. Note that the argument \mathbf{x} of $\tau_a^w(\mathbf{x})$ is a vector, while v_a in $t_a(v_a)$ is a scalar. We rewrite UE-PE- \mathbf{X} in (14) as follows:

$$\sum_{w \in \mathcal{W}} \sum_{a \in \mathcal{A}} \lambda_a^w \tau_a^w(\bar{\mathbf{x}})(x_a^w - \bar{x}_a^w) \geq 0 \quad \forall \mathbf{x} \in \mathbf{X} \quad (30)$$

for some $\lambda_a^w \in [\frac{1}{1+\kappa}, 1]$. It is straightforward to show that $\tau(\cdot)$ is monotone in \mathbf{X} ; however, it is not necessarily *strongly* monotone in \mathbf{X} . Since \mathbf{V} is a compact set, if $t(\mathbf{v})$ is a polynomial function, then $Z(\mathbf{v}) = \mathbf{t}(\mathbf{v})^\top \mathbf{v}$ is Lipschitz continuous in \mathbf{V} . That is

$$|Z(\mathbf{v}^1) - Z(\mathbf{v}^2)| \leq L \|\mathbf{v}^1 - \mathbf{v}^2\|_{\mathbf{V}}$$

for all $\mathbf{v}^1, \mathbf{v}^2 \in \mathbf{V}$ for some finite positive constant L .

Theorem 5. *Let $Z(\cdot)$ be Lipschitz continuous in \mathbf{V} with L and $\mathbf{t}(\cdot)$ be strongly monotone in \mathbf{V} with modulus α . Let \mathbf{x}^κ be a solution to (30) with κ , \mathbf{v}^κ be its corresponding flow vector in \mathbf{V} , and \mathbf{v}^0 be the PRUE flow vector. We have*

$$|Z(\mathbf{v}^\kappa) - Z(\mathbf{v}^0)| \leq \frac{\kappa L}{\alpha} \|\mathbf{t}(\mathbf{v}^0)\|_{\mathbf{V}}. \quad (31)$$

Proof of Theorem 5. See Appendix B. □

We can use Theorem 5 to provide both lower and upper bounds on $Z(\mathbf{v}^\kappa)$ in comparison to $Z(\mathbf{v}^0)$:

$$Z(\mathbf{v}^0) - \frac{\kappa L}{\alpha} \|\mathbf{t}(\mathbf{v}^0)\|_{\mathbf{V}} \leq Z(\mathbf{v}^\kappa) \leq Z(\mathbf{v}^0) + \frac{\kappa L}{\alpha} \|\mathbf{t}(\mathbf{v}^0)\|_{\mathbf{V}}.$$

The application of Theorem 5, however, is limited for two reasons. First, it depends on the Lipschitz constant L , which tends to be too large to provide tight bounds. Second, it assumes $\mathbf{t}(\cdot)$ to be strongly monotone in \mathbf{V} . It is well known that polynomial functions of order greater than 1 is not strongly monotone in \mathbb{R} . As shown in the counterexample in Appendix C, the travel time function $\mathbf{t}(\cdot)$, when it is a polynomial function, is not strongly monotone in \mathbf{V} as well. Therefore, Theorem 5 is only applicable to linear travel time functions.

5 Numerical Bounds

To quantify PoSat in typical traffic networks and compare it with the theoretical bound in Theorem 3, we define the worst-case problem for the total system travel time under MSatUE as follows:

$$\max \quad Z(\mathbf{v}^\kappa) = \sum_{a \in \mathcal{A}} z_a(v_a^\kappa) = \sum_{a \in \mathcal{A}} t_a(v_a^\kappa) v_a^\kappa \quad (32)$$

subject to \mathbf{v}^κ is an MSatUE flow with κ

To quantify the benefit of satisficing, we can minimize the objective function, instead of maximizing. Since MSatUE involves path-based definition and formulation, (32) is numerically more challenging to solve. Instead, we replace MSatUE by UE-PE. We know that the UE-PE models provide a subset of MSatUE traffic flow patterns as seen in Theorem 1; hence by using UE-PE models, we will obtain suboptimal solutions to (32).

Using UE-PE- \mathbf{X} in (12), we formulate the worst-case problem as follows:

$$\max \quad Z(\mathbf{v}^\kappa) = \sum_{a \in \mathcal{A}} z_a(v_a^\kappa) = \sum_{a \in \mathcal{A}} t_a(v_a^\kappa) v_a^\kappa \quad (33)$$

$$\text{subject to} \quad \sum_{a \in \mathcal{A}} \sum_{w \in \mathcal{W}} (t_a(v_a^\kappa) - \varepsilon_a^w) (x_a^w - x_a^{\kappa, w}) \geq 0 \quad \forall \mathbf{x} \in \mathbf{X} \quad (34)$$

$$v_a^\kappa = \sum_{w \in \mathcal{W}} x_a^{\kappa, w} \quad \forall a \in \mathcal{A} \quad (35)$$

$$0 \leq \varepsilon_a^w \leq \frac{\kappa}{1 + \kappa} t_a(v_a^\kappa) \quad \forall a \in \mathcal{A} \quad (36)$$

Problem (33) is an instance of mathematical programs with equilibrium constraints (MPEC). We can replace the equilibrium condition (34) by the following KKT conditions to create a single-level optimization problem:

$$t_a(v_a^\kappa) - \varepsilon_a^w + \pi_i^w - \pi_j^w \geq 0 \quad \forall w \in \mathcal{W}, a \in \mathcal{A} \quad (37)$$

$$x_a^{\kappa, w} (t_a(v_a^\kappa) - \varepsilon_a^w + \pi_i^w - \pi_j^w) = 0 \quad \forall w \in \mathcal{W}, a \in \mathcal{A} \quad (38)$$

$$\sum_{a \in \mathcal{A}_i^+} x_a^{\kappa, w} - \sum_{a \in \mathcal{A}_i^-} x_a^{\kappa, w} = q_i^w \quad \forall w \in \mathcal{W}, i \in \mathcal{N} \quad (39)$$

The resulting problem is a mathematical program with complementarity conditions (MPCC), which is nonlinear and nonconvex. Finding a global solution to MPCC problems is in general difficult, and Kleer and Schäfer (2016) has shown that solving the above MPCC optimally is NP-hard. We note that the problem is closely related to the worst-case problem proposed by Lou et al. (2010). While MPCC-based algorithms, as suggested by Lou et al. (2010), are valid for solving the MPCC problem, our intention is to use the sensitivity analysis for parametric variational inequalities by taking advantage of the form UE-PE- \mathbf{V} in (17).

5.1 Sensitivity Analysis in V

Letting $\tilde{t}_a(v_a, \lambda_a) = \lambda_a t_a(v_a)$, we consider, in vector form, the following parametric variational inequality problem:

$$\tilde{\mathbf{t}}(\bar{\mathbf{v}}, \bar{\boldsymbol{\lambda}})^\top (\mathbf{v} - \bar{\mathbf{v}}) \geq 0 \quad \forall \mathbf{v} \in \mathbf{V} \quad (40)$$

for any given $\bar{\boldsymbol{\lambda}}$. With the current parameter $\bar{\boldsymbol{\lambda}}$, the solution to (40) is $\bar{\mathbf{v}}$ —and $\bar{\mathbf{f}}$. As the parameter changes from $\bar{\boldsymbol{\lambda}}$ to $\bar{\boldsymbol{\lambda}} + \boldsymbol{\lambda}'$, the flow vectors $\bar{\mathbf{v}}$ and $\bar{\mathbf{f}}$ will change to $\bar{\mathbf{v}} + \mathbf{v}'$ and $\bar{\mathbf{f}} + \mathbf{f}'$, respectively. We want to estimate \mathbf{v}' and \mathbf{f}' , which will be used eventually to estimate $\frac{\partial Z}{\partial \lambda_a}$.

Patriksson (2004) shows that \mathbf{v}' may be characterized by the following system:

$$\mathbf{r}(\mathbf{v}', \boldsymbol{\lambda}')^\top (\mathbf{u} - \mathbf{v}') \geq 0 \quad \forall \mathbf{u} \in \mathbf{K} \quad (41)$$

where we define that

$$\mathbf{r}(\mathbf{v}', \boldsymbol{\lambda}') = \nabla_{\mathbf{v}} \tilde{\mathbf{t}}(\bar{\mathbf{v}}, \bar{\boldsymbol{\lambda}}) \mathbf{v}' + \nabla_{\boldsymbol{\lambda}} \tilde{\mathbf{t}}(\bar{\mathbf{v}}, \bar{\boldsymbol{\lambda}}) \boldsymbol{\lambda}' \quad (42)$$

$$\mathbf{K}(\bar{\mathbf{v}}, \bar{\boldsymbol{\lambda}}) = \mathbf{T}_{\mathbf{V}}(\bar{\mathbf{v}}) \cap \tilde{\mathbf{t}}(\bar{\mathbf{v}}, \bar{\boldsymbol{\lambda}})^\perp \quad (43)$$

$$\tilde{\mathbf{t}}(\bar{\mathbf{v}}, \bar{\boldsymbol{\lambda}})^\perp = \left\{ \mathbf{y} : \mathbf{y}^\top \tilde{\mathbf{t}}(\bar{\mathbf{v}}, \bar{\boldsymbol{\lambda}}) = 0 \right\} \quad (44)$$

$$\mathbf{T}_{\mathbf{V}}(\bar{\mathbf{v}}) \text{ is the tangent cone of the set } \mathbf{V} \text{ at } \bar{\mathbf{v}}, \quad (45)$$

where $\nabla_{\mathbf{v}} \tilde{\mathbf{t}}(\cdot, \cdot)$ and $\nabla_{\boldsymbol{\lambda}} \tilde{\mathbf{t}}(\cdot, \cdot)$ represent the Jacobian of $\tilde{\mathbf{t}}(\cdot, \cdot)$ with respect to \mathbf{v} and $\boldsymbol{\lambda}$, respectively.

Patriksson (2004) shows that $\mathbf{K}(\bar{\mathbf{v}}, \bar{\boldsymbol{\lambda}})$ can be equivalently written as the set of \mathbf{v}' such that:

$$\begin{aligned} v'_a &= \sum_{p \in \mathcal{P}} \delta_a^p f'_p && \forall a \in \mathcal{A} \\ \sum_{p \in \mathcal{P}_w} f'_p &= 0 && \forall w \in \mathcal{W} \\ f'_p &\begin{cases} \text{free,} & \text{if } \bar{f}_p^\kappa > 0 \\ \geq 0, & \text{if } \bar{f}_p^\kappa = 0, \tilde{c}_p(\bar{\mathbf{f}}, \bar{\boldsymbol{\lambda}}) = \tilde{\mu}_w(\bar{\mathbf{f}}, \bar{\boldsymbol{\lambda}}) \\ = 0, & \text{if } \bar{f}_p^\kappa = 0, \tilde{c}_p(\bar{\mathbf{f}}, \bar{\boldsymbol{\lambda}}) > \tilde{\mu}_w(\bar{\mathbf{f}}, \bar{\boldsymbol{\lambda}}). \end{cases} && \forall p \in \mathcal{P}_w, w \in \mathcal{W} \end{aligned}$$

where path travel time functions $\tilde{c}_p(\cdot, \cdot)$ and $\tilde{\mu}_w(\cdot, \cdot)$ are defined corresponding to arc travel time functions $\tilde{t}_a(\cdot, \cdot)$. Since the above representation requires path enumeration, we represent $\mathbf{K}(\bar{\mathbf{v}}, \bar{\boldsymbol{\lambda}})$ as the set of \mathbf{v}' such that:

$$v'_a = \sum_{w \in \mathcal{W}} x_a'^w \quad \forall a \in \mathcal{A} \quad (46)$$

$$\sum_{a \in \mathcal{A}_i^+} x_a'^w - \sum_{a \in \mathcal{A}_i^-} x_a'^w = 0 \quad \forall i \in \mathcal{N}, \forall w \in \mathcal{W} \quad (47)$$

$$x_a^{iw} \begin{cases} \text{free}, & \text{if } \bar{x}_a^w > 0 \\ \geq 0 & \text{if } \bar{x}_a^w = 0 \end{cases} \quad \forall w \in \mathcal{W}, i \in \mathcal{N} \quad (48)$$

$$\tilde{t}_a(\bar{v}_a, \bar{\lambda}_a)v'_a = 0 \quad \forall a \in \mathcal{A} \quad (49)$$

where (46)–(48) represent the tangent cone $\mathbf{T}_{\mathbf{V}}(\bar{\mathbf{v}})$ and (49) represents the orthogonal space (44).

With the strict monotonicity of $\tilde{t}(\cdot, \cdot)$, the solution \mathbf{v}' to (41) is unique, and it provides the directional derivative of \mathbf{v} with respect to $\boldsymbol{\lambda}$ at $(\bar{\mathbf{v}}, \bar{\boldsymbol{\lambda}})$ in direction of $\boldsymbol{\lambda}'$. We may obtain the solution to (41) by solving the following equivalent convex quadratic optimization problem (Josefsson and Patriksson, 2007):

$$\min_{\mathbf{v}'} [\nabla_{\mathbf{v}} \tilde{t}(\bar{\mathbf{v}}, \bar{\boldsymbol{\lambda}}) \boldsymbol{\lambda}']^\top \mathbf{v}' + \frac{1}{2} \sum_{a \in \mathcal{A}} \frac{\partial \tilde{t}_a(\bar{v}_a, \bar{\lambda}_a)}{\partial v_a} (v'_a)^2 \quad (50)$$

subject to (46)–(49)

for any given direction $\boldsymbol{\lambda}'$.

5.2 A Gradient Projection Algorithm

We replace the MSatUE requirement in (32) by UE-PE- \mathbf{V} in (17). Note that UE-PE- \mathbf{F} and UE-PE- \mathbf{X} may not be used in algorithms based on sensitivity analysis, since traffic equilibria in \mathbf{F} or \mathbf{X} are not unique in general; hence cannot guarantee the uniqueness of directional derivatives. With strict monotonicity of arc travel time functions $t_a(\cdot)$, we can assure the uniqueness as discussed in the above.

We consider

$$\max_{\boldsymbol{\lambda}} Z = \sum_{a \in \mathcal{A}} t_a(v_a^\kappa) v_a^\kappa \quad (51)$$

$$\text{subject to } \sum_{a \in \mathcal{A}} \lambda_a t_a(v_a^\kappa) (v_a - v_a^\kappa) \geq 0 \quad \forall \mathbf{v} \in \mathbf{V} \quad (52)$$

$$\frac{1}{1 + \kappa} \leq \lambda_a \leq 1 \quad \forall a \in \mathcal{A} \quad (53)$$

$$\mathbf{v}^\kappa \in \mathbf{V} \quad (54)$$

To approximate $\frac{\partial \mathbf{v}}{\partial \lambda_a}$ using the sensitivity problem (50), we consider arc $a \in \mathcal{A}$ and we construct a direction $\boldsymbol{\lambda}'$ with each element being

$$\lambda'_e = \begin{cases} 1 & \text{if } e = a \\ 0 & \text{if } e \neq a, e \in \mathcal{A} \end{cases}$$

Given this direction $\boldsymbol{\lambda}'$, we solve (50) to obtain \mathbf{v}' , which approximates $\frac{\partial \mathbf{v}^\kappa}{\partial \lambda_a}$. We repeat this procedure for all $a \in \mathcal{A}$.

Once we obtained the Jacobian matrix $\nabla_{\lambda} \mathbf{v}$, we compute the gradient necessary for solving (51) as follows:

$$\nabla_{\lambda} Z = \nabla_{\lambda} \mathbf{v} \nabla_{\mathbf{v}} Z.$$

When the incumbent solution is $\bar{\lambda}$ and its corresponding equilibrium solution is $\bar{\mathbf{v}}$, we have the following projection rule to solve (51) iteratively:

$$\lambda^{\text{new}} = \left[\bar{\lambda} + \theta \nabla_{\lambda} Z \Big|_{\lambda=\bar{\lambda}, \mathbf{v}=\bar{\mathbf{v}}} \right]_{\frac{1}{1+\kappa}}^1$$

with an appropriate step size θ , where $[x]_a^b$ is the projection on to the interval $[a, b]$, i.e. $\max(a, \min(x, b))$.

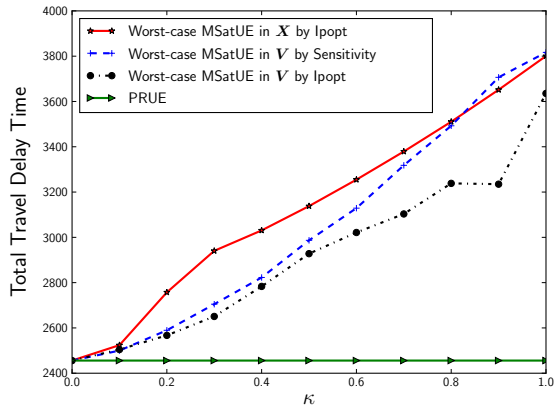
5.3 Numerical Experiments

We present some examples to compare the total travel times in MSatUE and PRUE numerically and compare the numerical worst-cases with the theoretical bound given in Theorem 3. We also test the performance of the sensitivity-based gradient projection algorithm for solving problem (51), which is the worst-case MSatUE problem with UE-PE- \mathbf{V} . The benchmarks are solutions to (32) defined with UE-PE- \mathbf{X} and UE-PE- \mathbf{V} , solved by the Ipopt nonlinear solver (Wächter and Biegler, 2006), after reformulating as a single-level optimization problem using KKT conditions. We use the Julia Language and the JuMP package (Dunning et al., 2017) for modeling and interfacing with the Ipopt solver.

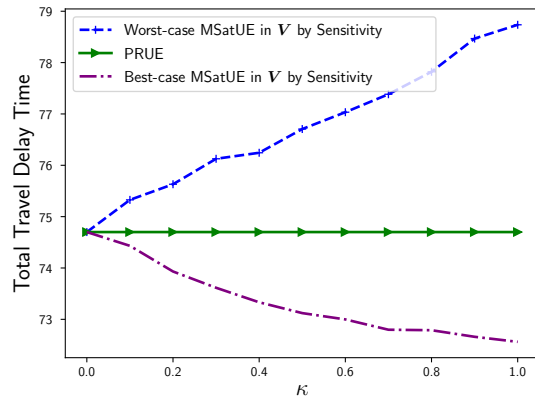
First, we consider the nine-node network presented in Hearn and Ramana (1998). The nine-node network consists of 9 nodes and 18 arcs, and the travel time functions are polynomials of order $n = 4$. The comparison result is presented in Figure 4a. As (32) is a non-convex problem, the Ipopt solver can produce a local minimum at best. The sensitivity-based algorithm certainly outperforms the Ipopt solver in \mathbf{V} , since the sensitivity-based algorithm utilizes the specific structure of parametric variational inequalities. On the other hand, the Ipopt solver in \mathbf{X} outperforms the sensitivity-based algorithm in \mathbf{V} , which is as expected from Lemma 3. For bigger values of κ , however, we observe that the sensitivity-based algorithm works better. This can happen, because both Ipopt and the sensitivity-based algorithm find local optima and the sensitivity-based algorithm can find a better one.

We also consider the Sioux Falls network presented in Suwansirikul et al. (1987). Sioux Falls is a network with 24 nodes and 76 arcs and 576 OD pairs with $n = 4$. Due to the large size of the Sioux Falls network problems and the difficulty of the problems, the Ipopt solver takes more than 24 hours to solve the Sioux Falls network problems, unless it generates errors and fails to find a solution. We only present our results obtained by the sensitivity-based algorithm. Figure 4b represents the total travel time in the case of PRUE, and the best- and worst-cases of MSatUE. Since we only consider \mathbf{V} in the sensitivity-based algorithm, the actual bounds for the best- and worst-cases can be smaller and larger, respectively.

Figure 5a compares the total travel time in the worst-case scenario with the theoretical bound

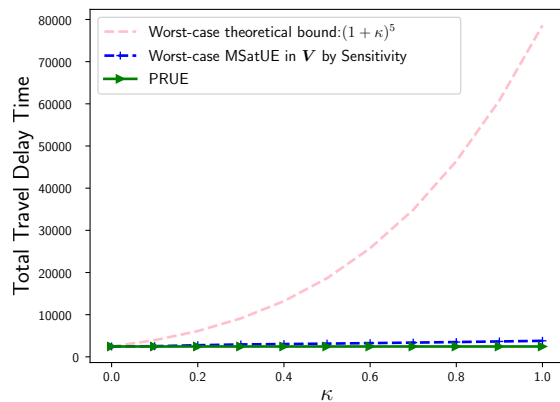


(a) The nine-node network of Hearn and Ramana (1998)

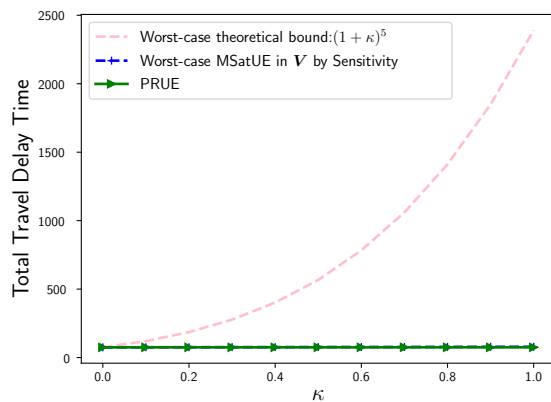


(b) The Sioux Falls network

Figure 4: Comparing total travel times



(a) The nine-node network of Hearn and Ramana (1998)



(b) The Sioux Falls network

Figure 5: Comparing with the theoretical bound

provided in Theorem 3 for MSatUE for the nine-node network; similarly Figure 5b for the Sioux Falls network. We observe that there is a large gap between the theoretical and numerical bounds. Although the theoretical result certainly provides a valid bound, it is too large to be practically useful in realistic road networks. This indicates opportunities for empirical studies on the bounds and other theoretical bounds that depend on more network-specific information such as travel demands and travel time functions. The bound $(1 + \kappa)^n$ in Theorem 3 is independent from such network-specific information.

6 Concluding Remarks

When network users are satisficing decision makers, the resulting satisficing user equilibria may lose the system performance, compared to the perfectly rational user equilibrium. To quantify how much we can lose, this paper has quantified the worst-case theoretical bound on the price of satisficing and suggested a numerical algorithm based on sensitivity analysis.

We suggest potential future research directions both in theoretical and numerical bounds. For the theoretical bound, our result is based on the condition (25). By attempting to relax this condition, one may obtain a global bound for any value of κ .

To improve the numerical bounds, we need to consider the problem in \mathbf{X} , instead of \mathbf{V} as done in this paper. The critical challenge of applying the sensitivity analysis results in \mathbf{X} is that solutions to the equilibrium problems in \mathbf{X} are not unique in general; hence differentiability with respect to perturbations cannot be guaranteed. Algorithms without requiring differentiability are necessary to solve the problems in \mathbf{X} .

In deriving the theoretical bounds, we utilized a novel technique comparing equilibrium patterns before and after the travel demand is increased; namely \mathbf{V} and $\mathbf{V}_{1+\kappa}$. Applying this technique in the context of the burden of risk aversion and the deviation ratio would be an interesting research direction.

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Appendices

A Comparison of System Optimal and BRUE Flows

Lemma 8. *Let $\mathbf{f}^* \in \mathbf{F}$ and $\hat{\mathbf{f}}^* \in \mathbf{F}_{1+\kappa}$ be the system optimal flow vectors, and $\mathbf{f}^0 \in \mathbf{F}$ and $\hat{\mathbf{f}}^0 \in \mathbf{F}_{1+\kappa}$ be PRUE flow vectors with the corresponding travel demands. Suppose that $t_a(\cdot)$ are monotone for all $a \in \mathcal{A}$. Then we have*

$$(1 + \kappa)C(\mathbf{f}^*) \leq C(\hat{\mathbf{f}}^*) \quad (55)$$

$$(1 + \kappa)C(\mathbf{f}^0) \leq C(\hat{\mathbf{f}}^0) \quad (56)$$

for all $\kappa \geq 0$.

Proof of Lemma 8. 1. We consider corresponding arc flow vectors \mathbf{v}^* and $\hat{\mathbf{v}}^*$. and let $\bar{\mathbf{v}} = \frac{\hat{\mathbf{v}}^*}{1+\kappa}$ so that $\bar{\mathbf{v}} \in \mathbf{V}$. Then

$$C(\hat{\mathbf{f}}^*) = Z(\hat{\mathbf{v}}^*) = \sum_{a \in \mathcal{A}} t_a((1 + \kappa)\bar{v}_a)(1 + \kappa)\bar{v}_a$$

$$\begin{aligned}
&\geq (1 + \kappa) \sum_{a \in \mathcal{A}} t_a((1 + \kappa)\bar{v}_a)\bar{v}_a \\
&\geq (1 + \kappa) \sum_{a \in \mathcal{A}} t_a(\bar{v}_a)\bar{v}_a \\
&\geq (1 + \kappa) \min_{\mathbf{v} \in \mathbf{V}} \sum_{a \in \mathcal{A}} t_a(v_a)v_a \\
&= (1 + \kappa) \min_{\mathbf{f} \in \mathbf{F}} \sum_{p \in \mathcal{P}} c_p(\mathbf{f})f_p \\
&= (1 + \kappa)C(\mathbf{f}^*).
\end{aligned}$$

2. From the monotonicity of $t_a(\cdot)$ and Theorem 4.2 of Dafermos and Nagurney (1984), we have

$$\sum_{w \in \mathcal{W}} (\mu_w(\hat{\mathbf{f}}^0) - \mu_w(\mathbf{f}^0))(\hat{Q}_w - Q_w) \geq 0 \tag{57}$$

where $\hat{Q} = (1 + \kappa)Q$. Therefore,

$$\begin{aligned}
0 &\leq \sum_{w \in \mathcal{W}} (\mu_w(\hat{\mathbf{f}}^0) - \mu_w(\mathbf{f}^0))((1 + \kappa)Q_w - Q_w) \\
&= \kappa \sum_{w \in \mathcal{W}} (\mu_w(\hat{\mathbf{f}}^0) - \mu_w(\mathbf{f}^0))Q_w \\
&= \frac{\kappa}{1 + \kappa} \sum_{w \in \mathcal{W}} \mu_w(\hat{\mathbf{f}}^0)(1 + \kappa)Q_w - \kappa \sum_{w \in \mathcal{W}} \mu_w(\mathbf{f}^0)Q_w \\
&= \frac{\kappa}{1 + \kappa} \sum_{w \in \mathcal{W}} \mu_w(\hat{\mathbf{f}}^0)\hat{Q}_w - \kappa \sum_{w \in \mathcal{W}} \mu_w(\mathbf{f}^0)Q_w \\
&= \frac{\kappa}{1 + \kappa} C(\hat{\mathbf{f}}^0) - \kappa C(\mathbf{f}^0)
\end{aligned}$$

Hence we obtain the lemma. □

B Proof of Theorem 5

To prove Theorem 5, we need a few lemmas first. For notational simplicity, we let $\tilde{\tau}_a^w(x) = \lambda_a^w \tau_a(x)$.

Lemma 9. *Assume $\mathbf{t}(\cdot)$ is strongly monotone in \mathbf{V} with modulus α . Let \mathbf{v}^1 and \mathbf{v}^2 be two arbitrary flow vectors in \mathbf{V} , and \mathbf{x}^1 and \mathbf{x}^2 be the corresponding vectors in \mathbf{X} . Then we have*

$$[\tilde{\tau}(\mathbf{x}^1) - \tilde{\tau}(\mathbf{x}^2)]^\top (\mathbf{x}^1 - \mathbf{x}^2) \geq \frac{\alpha}{1 + \kappa} \|\mathbf{v}^1 - \mathbf{v}^2\|_{\mathbf{V}}^2$$

for all $\mathbf{x}^1, \mathbf{x}^2 \in \mathbf{X}$.

Proof of Lemma 9.

$$[\tilde{\tau}(\mathbf{x}^1) - \tilde{\tau}(\mathbf{x}^2)]^\top (\mathbf{x}^1 - \mathbf{x}^2) = \sum_{w \in \mathcal{W}} \sum_{a \in \mathcal{A}} \lambda_a^w [\tilde{\tau}_a^w(x^1) - \tilde{\tau}_a^w(x^2)]^\top (x_a^{1w} - x_a^{2w})$$

$$\begin{aligned}
&\geq \frac{1}{\kappa + 1} \sum_{w \in \mathcal{W}} \sum_{a \in \mathcal{A}} [\tilde{\tau}_a^w(x^1) - \tilde{\tau}_a^w(x^2)]^\top (x_a^{1w} - x_a^{2w}) \\
&= \frac{1}{\kappa + 1} \sum_{a \in \mathcal{A}} [t_a(v_a^1) - t_a(v_a^2)](v_a^1 - v_a^2) \\
&= \frac{1}{1 + \kappa} [\mathbf{t}(\mathbf{v}^1) - \mathbf{t}(\mathbf{v}^2)]^\top (\mathbf{v}^1 - \mathbf{v}^2) \\
&\geq \frac{\alpha}{1 + \kappa} \|\mathbf{v}^1 - \mathbf{v}^2\|_{\mathbf{V}}^2
\end{aligned}$$

□

Lemma 10. Let \mathbf{v}^κ be a solution to (30) with κ , and \mathbf{v}^0 be perfect user equilibrium flow vector. Then

$$[\boldsymbol{\tau}(x^0) - \tilde{\boldsymbol{\tau}}(x^0)]^\top (\mathbf{x}^\kappa - \mathbf{x}^0) \leq \frac{\kappa}{1 + \kappa} \mathbf{t}(\mathbf{v}^0)^\top (\mathbf{v}^\kappa - \mathbf{v}^0)$$

Proof of 10. Note that

$$\begin{aligned}
[\boldsymbol{\tau}(x^0) - \tilde{\boldsymbol{\tau}}(x^0)]^\top (\mathbf{x}^\kappa - \mathbf{x}^0) &= \sum_a \sum_w (1 - \lambda_a^w) t_a(v_a^0) (v_a^\kappa - v_a^w) \\
&\leq \frac{\kappa}{1 + \kappa} \sum_a \sum_w t_a(v_a^0) (v_a^\kappa - v_a^w) \\
&= \frac{\kappa}{1 + \kappa} \mathbf{t}(\mathbf{v}^0)^\top (\mathbf{v}^\kappa - \mathbf{v}^0)
\end{aligned}$$

since $\frac{1}{1 + \kappa} \leq \lambda_a^w \leq 1$. □

Lemma 11. Let \mathbf{v}^κ be a solution to (30) with κ , and \mathbf{v}^0 be a perfect user equilibrium flow vector. When $\mathbf{t}(\cdot)$ is strongly monotone in \mathbf{V} with modulus α , then

$$\|\mathbf{v}^\kappa - \mathbf{v}^0\|_{\mathbf{V}} \leq \frac{\kappa}{\alpha} \|\mathbf{t}(\mathbf{v}^0)\|_{\mathbf{V}}$$

Proof of 11. Since x^0 is at a user equilibrium, we have $\boldsymbol{\tau}(x^0)^\top (\mathbf{x} - x^0) \geq 0$ for all $\mathbf{x} \in \mathbf{X}$. Letting $\mathbf{x} = \mathbf{x}^\kappa$, we obtain

$$\boldsymbol{\tau}(x^0)^\top (\mathbf{x}^\kappa - \mathbf{x}^0) \geq 0. \quad (58)$$

Similarly, since \mathbf{x}^κ is a solution to (30) and $\mathbf{x}^0 \in \mathbf{X}$, we have

$$\tilde{\boldsymbol{\tau}}(\mathbf{x}^\kappa)^\top (\mathbf{x}^0 - \mathbf{x}^\kappa) \geq 0. \quad (59)$$

By adding the both sides of (58) and (59), we obtain

$$[\tilde{\boldsymbol{\tau}}(\mathbf{x}^\kappa) - \boldsymbol{\tau}(x^0)]^\top (\mathbf{x}^\kappa - \mathbf{x}^0) \leq 0. \quad (60)$$

By adding and subtracting $\tilde{\boldsymbol{\tau}}(x^0)$ in (60), we have

$$[\tilde{\boldsymbol{\tau}}(\mathbf{x}^\kappa) - \boldsymbol{\tau}(x^0) + \tilde{\boldsymbol{\tau}}(x^0) - \tilde{\boldsymbol{\tau}}(x^0)]^\top (\mathbf{x}^\kappa - \mathbf{x}^0) \leq 0$$

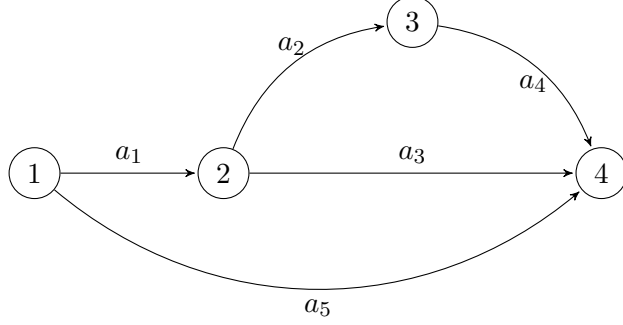


Figure 6: A counterexample for strong monotonicity. Travel demand is 1 from node 1 to node 4.

and by rearranging the terms, we obtain

$$[\tilde{\boldsymbol{\tau}}(\mathbf{x}^\kappa) - \tilde{\boldsymbol{\tau}}(\mathbf{x}^0)]^\top (\mathbf{x}^\kappa - \mathbf{x}^0) \leq [\boldsymbol{\tau}(\mathbf{x}^0) - \tilde{\boldsymbol{\tau}}(\mathbf{x}^0)]^\top (\mathbf{x}^\kappa - \mathbf{x}^0). \quad (61)$$

Using Lemmas 9 and 10 with (61), we obtain

$$\begin{aligned} \frac{\alpha}{1+\kappa} \|\mathbf{v}^\kappa - \mathbf{v}^0\|_{\mathbf{V}}^2 &\leq [\tilde{\boldsymbol{\tau}}(\mathbf{x}^\kappa) - \tilde{\boldsymbol{\tau}}(\mathbf{x}^0)]^\top (\mathbf{x}^\kappa - \mathbf{x}^0) \\ &\leq [\boldsymbol{\tau}(\mathbf{x}^0) - \tilde{\boldsymbol{\tau}}(\mathbf{x}^0)]^\top (\mathbf{x}^\kappa - \mathbf{x}^0) \\ &\leq \frac{\kappa}{1+\kappa} \mathbf{t}(\mathbf{v}^0)^\top (\mathbf{v}^\kappa - \mathbf{v}^0). \end{aligned}$$

Therefore, we have

$$\frac{\alpha}{1+\kappa} \|\mathbf{v}^\kappa - \mathbf{v}^0\|_{\mathbf{V}}^2 \leq \frac{\kappa}{1+\kappa} \|\mathbf{t}(\mathbf{v}^0)\|_{\mathbf{V}} \|\mathbf{v}^\kappa - \mathbf{v}^0\|_{\mathbf{V}}$$

which leads to the lemma. \square

From the Lipschitz continuity of $Z(\cdot)$ in \mathbf{V} and Lemma 11, we obtain Theorem 5.

C A Counterexample for Strong Monotonicity of $\mathbf{t}(\cdot)$ in \mathbf{V}

Consider the network in Figure 6. Suppose that the cost function for each arc is $t_a(v_a) = (v_a)^n$ for $n > 1$. To see if $\mathbf{t}(\cdot)$ is strongly monotone in \mathbf{V} , we need to find a positive constant α such that

$$\sum_{a \in \mathcal{A}} [t_a(v_a^1) - t_a(v_a^2)](v_a^1 - v_a^2) \geq \alpha \sum_{a \in \mathcal{A}} (v_a^1 - v_a^2)^2 \quad (62)$$

for all $\mathbf{v}^1, \mathbf{v}^2 \in \mathbf{V}$. Suppose $\mathbf{v}^1 = (\varepsilon, \varepsilon, 0, \varepsilon, 1 - \varepsilon)$ and $\mathbf{v}^2 = (\varepsilon, 0, \varepsilon, 0, 1 - \varepsilon)$ represent two arc flow vectors satisfying the travel demand 1. The left-hand-side of (62) is $3\varepsilon^{n+1}$ and the right-hand-side is $3\varepsilon^2$. Consequently, we need

$$3\varepsilon^{n+1} \geq 3\alpha\varepsilon^2.$$

By making $\varepsilon \rightarrow 0$, we observe that $\alpha \rightarrow 0$; hence non-zero α does not exist. Therefore, in general, $\mathbf{t}(\cdot)$ is not a strongly monotone function in \mathbf{V} as well as in $\mathbb{R}^{|\mathcal{A}|}$.