# Sensitivity of Wardrop Equilibria: Revisited

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**Abstract**

For *single-commodity* networks, Englert et al. [*Theory Comput Syst*, **47**(1):3–14 (2010)] have shown that the increase of the price of anarchy is bounded by a factor of $(1+\varepsilon)^p$ from above, when the travel demand is increased by a factor of $1+\varepsilon$ and the latency functions are polynomials of degree at most $p$. We show that the same upper bound holds for *multi-commodity* networks and provide a lower bound as well.

**Keywords** Wardrop equilibria; Selfish routing; Sensitivity analysis

## 1 Introduction and Notation

We study Wardrop's traffic equilibria (Wardrop, 1952) and how the price of anarchy changes with demand increases. For *single-commodity* networks, Englert et al. (2010) have provided the upper bound on the increase of the price of anarchy when the demand increases. In this paper, we show that the same bound is also valid for *multi-commodity* networks. We also provide a lower bound when the price of anarchy decreases as the demand increases.

In this paper, we follow the notation used in Englert et al. (2010). For a given directed graph $G = (V, E)$, we consider non-decreasing latency functions $\ell_e : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ for each edge $e \in E$. For each commodity $i \in [k] = \{1, 2, ..., k\}$, the flow demand is $d_i$. We let $\mathcal{P}_i$ denote the available paths for commodity $i$ and $\mathcal{P} = \cup_{i\in[k]}\mathcal{P}_i$. Let $(G, (d_i), \ell)$ denote an instance of Wardrop equilibrium problems.

A feasible path flow vector $f$ is feasible when $\sum_{P\in\mathcal{P}_i} f_P = d_i$ for all $i \in [k]$ and $f_p \geq 0$ for all $p \in \mathcal{P}$. A path flow vector $f$ can also be written for each edge $e$, such that $f_e = \sum_{i\in[k]} \sum_{P\in\mathcal{P}_i:e\in P} f_P$. The path latency is defined as $\ell_P(f) = \sum_{e\in P} \ell_e(f_e)$. The total cost is defined as $C(f) = \sum_{P\in\mathcal{P}} \ell_P(f) f_P = \sum_{e\in E} \ell_e(f_e) f_e$.

For each commodity $i \in [k]$, we define

$$\mu_i(f) = \min_{P\in\mathcal{P}_i} \ell_P(f_P).$$

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We consider the latency function for each edge $e \in E$ of the following form:

$$\ell_e(f_e) = \sum_{m=0}^{p} b_{em}(f_e)^m$$

for constants $b_{em} \geq 0$ for all $e \in E$ and $m = 0, 1, ..., p$.

**Definition 1.** A feasible flow vector $f$ is at Wardrop equilibrium if

$$f_P > 0 \implies \ell_P(f) = \mu_i(f)$$

for all $P \in \mathcal{P}_i$ and $i \in [k]$.

## 2 Changes of the Price of Anarchy

We consider the demand changes from $d_i$ to $(1+\varepsilon)d_i$ for all commodity $i \in [k]$ for some $\varepsilon \geq 0$. Although Theorem 3 of Englert et al. (2010) considers single-commodity networks and focuses on path latency, the same technique is valid for showing the following theorem for multi-commodity networks. While the bound on the path latency does not hold in multi-commodity networks as noted by Englert et al. (2010), it still provides a bound on the total cost. Using the result of Dafermos and Nagurney (1984), we can also provide a lower bound.

**Theorem 1.** *Let $C_{\text{opt}}$ and $C'_{\text{opt}}$ be the cost of an optimal flow for instances $(G, (d_i), \ell)$ and $(G, ((1+\varepsilon)d_i), \ell)$ with polynomial latency functions of degree at most $p$ with nonnegative coefficients, respectively, and let $f$ and $f'$ be equilibrium flows, respectively. Then we can show*

$$(1+\varepsilon)C_{\text{opt}} \leq C'_{\text{opt}} \leq (1+\varepsilon)^{p+1}C_{\text{opt}}, \tag{1}$$

$$(1+\varepsilon)C(f) \leq C(f') \leq (1+\varepsilon)^{p+1}C(f). \tag{2}$$

*Proof.* 1. We let $f^{*\prime}$ be the optimal flow for instance $(G, ((1+\varepsilon)d_i), \ell)$ and $\overline{f} = \frac{f^{*\prime}}{1+\varepsilon}$. Then,

$$\begin{aligned} C'_{\text{opt}} &= \sum_{e \in E} \ell_e((1+\varepsilon)\overline{f}_e)(1+\varepsilon)\overline{f}_e \\ &= (1+\varepsilon) \sum_{e \in E} \ell_e((1+\varepsilon)\overline{f}_e)\overline{f}_e \\ &\geq (1+\varepsilon) \sum_{e \in E} \ell_e(\overline{f}_e)\overline{f}_e \\ &\geq (1+\varepsilon)C_{\text{opt}}. \end{aligned}$$

Also,

$$C'_{\text{opt}} \leq \sum_{e \in E} \ell_e((1+\varepsilon)f_e)(1+\varepsilon)f_e$$

$$
\begin{aligned}
&= \sum_{e\in E}\left(\sum_{m=0}^{p} b_{em}((1+\varepsilon)f_e)^m\right)(1+\varepsilon)f_e \\
&\le \sum_{e\in E}\left(\sum_{m=0}^{p} b_{em}(1+\varepsilon)^p(f_e)^m\right)(1+\varepsilon)f_e \\
&\le (1+\varepsilon)^{p+1}\sum_{e\in E}\ell_e(f_e)f_e \\
&= (1+\varepsilon)^{p+1}C_{\mathrm{opt}}.
\end{aligned}
$$

2. From the monotonicity of $\ell_e(\cdot)$ and Theorem 4.2 of Dafermos and Nagurney (1984), we have

$$
\sum_{i\in[k]}(\mu_i(f')-\mu_i(f))(d_i'-d_i)\ge 0
$$

where $d_i'=(1+\varepsilon)d_i$. Therefore,

$$
\begin{aligned}
0 &\le \sum_{i\in[k]}(\mu_i(f')-\mu_i(f))((1+\varepsilon)d_i-d_i) \\
&= \varepsilon\sum_{i\in[k]}(\mu_i(f')-\mu_i(f))d_i \\
&= \frac{\varepsilon}{1+\varepsilon}\sum_{i\in[k]}\mu_i(f')(1+\varepsilon)d_i-\varepsilon\sum_{i\in[k]}\mu_i(f)d_i \\
&= \frac{\varepsilon}{1+\varepsilon}\sum_{i\in[k]}\mu_i(f')d_i'-\varepsilon\sum_{i\in[k]}\mu_i(f)d_i \\
&= \frac{\varepsilon}{1+\varepsilon}C(f')-\varepsilon C(f),
\end{aligned}
$$

which leads to $(1+\varepsilon)C(f)\le C(f')$.

The remaining inequality, $C(f')\le(1+\varepsilon)^{p+1}C(f)$, is already proved by the proof of Theorem 3 in Englert et al. (2010). □

When the demand increases, we can observe that the cost of both the optimal flow and the equilibrium flow increases at least by factor of $1+\varepsilon$.

We obtain the following result:

**Theorem 2.** *Let $\rho$ and $\rho'$ denote the Price of Anarchy for instances $(G,(d_i),\ell)$ and $(G,((1+\varepsilon)d_i),\ell)$ with polynomial latency functions of degree at most $p$ with nonnegative coefficients, respectively. Then $\frac{1}{(1+\varepsilon)^p}\cdot\rho\le\rho'\le(1+\varepsilon)^p\cdot\rho$.*

*Proof.* We can show that

$$
\frac{\rho'}{\rho}=\frac{C(f')/C'_{\mathrm{opt}}}{C(f)/C_{\mathrm{opt}}}=\frac{C(f')}{C(f)}\cdot\frac{C_{\mathrm{opt}}}{C'_{\mathrm{opt}}}\le(1+\varepsilon)^{p+1}\cdot\frac{1}{1+\varepsilon}=(1+\varepsilon)^p
$$

where the inequality holds by Theorem 1. Similarly,

$$\frac{\rho}{\rho'} = \frac{C(f)/C_{\text{opt}}}{C(f')/C'_{\text{opt}}} = \frac{C(f)}{C(f')} \cdot \frac{C'_{\text{opt}}}{C_{\text{opt}}} \leq \frac{1}{1+\varepsilon} \cdot (1+\varepsilon)^{p+1} = (1+\varepsilon)^{p}$$

□

The upper bound is identical to the result of Englert et al. (2010), but holds for multi-commodity networks. O'Hare et al. (2016) study how the price of anarchy may decay as the demand increases. When the price of anarchy decreases, the lower bound in Theorem 2 provides useful information.

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