Worst-case Conditional Value-at-Risk Minimization for Hazardous Materials Transportation

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Despite significant advances in risk management, routing hazardous materials (hazmat) has relied on relatively simpler methods. In this paper, we formally introduce an advanced risk measure, called conditional value-at-risk (CVaR), applied to truck routing problems for hazmat transportation. We find that CVaR offers a flexible, risk-averse, and computationally tractable routing method that is adequate to mitigate hazmat accidents. We further extend CVaR to consider the worst-case CVaR (WCVaR) under data uncertainty. The two important data types in hazmat transportation are accident probabilities and accident consequences, both of which are subject to many ambiguous factors. In addition, historical data are usually insufficient to construct probability distribution of accident probabilities and consequences. This motivates a new robust optimization approach to consider and compute WCVaR. Important axioms are studied for both CVaR and WCVaR risk measures to be coherent and appropriate in the context of hazmat transportation, and computational methods are proposed. We demonstrate the proposed notions of CVaR and WCVaR through a case study in a realistic road network.

Key words: hazardous materials transportation; conditional value-at-risk; robust optimization

1. Introduction

The Pipeline and Hazardous Materials Safety Administration (2013) defines hazardous materials (hazmat) as “a substance or material capable of posing an unreasonable risk to health, safety, or property when transported in commerce”, and the US Department of Transportation manages hazmat transportation with nine classifications (Federal Motor Carrier Safety Administration 2001). Hazmat accidents can result in significant injuries to the population and damages to the environment. In 2007, hazmat shipments accounted for about 18 percent of the total freight shipped in the U.S. on a tonnage basis (U.S. Census Bureau 2007). Moreover, the volume has been increasing by about 5 percent every year (Transportation Research Board 2005).

Trucks are the most widely used mode for hazmat transportation for relatively short distances. The popularity of trucks for short-haul hazmat transport stems from their flexibility in operations, i.e., their ability to pick up and drop off hazmat close to its point of origin and destination,
respectively. As might be expected, the number of hazmat incidents involving the truck transport mode is also the highest. While all modes can cause severe consequences to our community, trucks have a more direct impact on our safety: people could be seriously injured or killed, important infrastructure systems could be damaged, residential environments could be destroyed, and so on.

In hazmat transportation, the extreme consequences of accidents must be avoided. However, the current routing method relies on simple rules or the most economical route, despite the significant advances in the risk management research field. In addition, the existing routing methods reported in the literature do not provide risk-averse routing, or lack the flexibility that is necessary to accommodate various practical factors. To fill the gap, new risk measures and routing methods based on conditional value-at-risk (CVaR) are proposed.

We claim that CVaR has a potential to be an important risk measure and decision-making tool in the context of hazmat transportation with low-probability-high-consequence nature. In particular, we emphasize the following: (1) CVaR is a proper and theoretically sound risk measure in hazmat transportation; (2) CVaR provides a flexible decision-making tool; (3) CVaR is an averse risk measure that focuses on large consequences; and (4) CVaR provides a computationally tractable optimization framework.

The CVaR concept is closely related to the notion of value-at-risk (VaR). The concept of VaR has been widely applied in the financial and economic fields (Duffie and Pan 1997, Linsmeier and Pearson 2000). VaR was originally introduced to measure risk in highly diversified financial investments. VaR was questioned as a proper risk measure not only because it is not coherent in the sense of Artzner et al. (1999), but also because it is possible to lead in an inaccurate perception of risk (Nocera 2009, Einhorn 2008). Critics claim that VaR as a risk measure ignores and cuts off what would happen in the distribution’s tail. While both VaR and CVaR are quantile-based risk measures, the main difference is that CVaR focuses in the long tail of the distribution where VaR on the other hand does not. This provides a motivation for CVaR in hazmat transportation.

The notion of CVaR is linked to the notions of Expected Tail Loss, Tail Conditional Expectation, Tail VaR, Average VaR, Worst Conditional Expectation, or Expected Shortfall (Rockafellar and Uryasev 2000, Dowd and Blake 2006) and some are equivalent. However, depending on the precise mathematical definitions and the characteristics of the associated random risk variable (especially when discontinuous), the meanings of the above names can be different (Acerbi and Tasche 2002, Sarykalin et al. 2008). CVaR is shown to be a coherent risk measure in the sense of Artzner et al. (1999)—satisfying translation invariance, subadditivity, positive homogeneity, and monotonicity—for general loss distributions, as well as for discrete distributions (Pflug 2000, Rockafellar and Uryasev 2002).
CVaR in hazmat transportation provides a *risk-averse* tool for decision making. Even though most of the existing hazmat routing methods, take the entire risk distribution under consideration, CVaR clearly offers more focus on the distribution’s long tail avoiding extreme events. In financial investment problems, focusing solely on the long tail in some cases may not be optimal, since high risk can result high return. Nevertheless, in the concept of hazmat transportation the return is finishing the transportation job at minimal or reasonable cost, and hazmat accidents are often catastrophic; therefore high-risk in general cannot be traded for high-return. Being risk-averse by focusing on the long tail is thus more reasonable.

Furthermore, CVaR provides a computationally tractable optimization framework. In financial portfolio management, CVaR optimization may be solved by a linear programming problem for both continuous and discrete random variables, with sampling or as it is (Rockafellar and Uryasev 2000, Mansini et al. 2007). CVaR in hazmat transportation, however, involves binary variables; therefore, there is added complexity in the problem solving. This paper proposes a tractable computational method.

However, there is a critical issue that needs to be addressed in any hazmat routing method: data uncertainty. While the consequences of hazmat accidents usually cause serious problems, the hazmat-accident data are usually insufficient to construct probability distributions. The accident probability of hazmat trucks in each road segment is difficult to estimate and consequences from such accidents are dependent on various uncertain factors such as severity of the accident, weather conditions at the time of the accident, and the quantity of hazmat release (Kwon et al. 2013). This property makes any hazmat routing method based on the stochastic assumptions on the data less meaningful.

In this paper, we propose a robust optimization framework for the routing methods based on the CVaR risk measure, assuming that data are uncertain within given sets. The proposed robust optimization method is closely related to robust shortest path (RSP) problems, which find a path that minimizes the worst-case travel cost with an uncertain set of travel cost data. When the uncertain set is box-constrained, the RSP problem can be solved in polynomial time (Bertsimas and Sim 2003), while the problem is NP-hard when the uncertain set is an ellipsoid (Bertsimas and Sim 2004, Chaerani et al. 2005) and a set of scenarios (Kouvelis and Yu 1996). We refer readers to Ben-Tal et al. (2009) and Gabrel et al. (2012) and references therein for general robust optimization methods.

In particular, we consider a worst-case CVaR (WCVaR) minimization problem for hazmat routing. For financial portfolio optimization applications, worst-cases have been considered for VaR (El Ghaoui et al. 2003) and CVaR (Zhu and Fukushima 2009). Our worst-case problem is structurally different from problems that are previously studied in the literature. There are two
types of uncertain data in hazmat problems: accident probability and accident consequence. Since the sources for these two types of data are generally different, the level of uncertainty in each data type is independent from each other and it is difficult to be considered as a single uncertain data (Kwon et al. 2013). We base our methodological developments on the robust shortest path problem with two uncertain multiplicative cost coefficients as in Kwon et al. (2013).

We distinguish the two kinds of robustness: one from the CVaR concept itself, and the other from the worst-case consideration in WCVaR. CVaR, especially with high probability threshold, provides protections against high loss in the underlying risk; therefore, CVaR provides “robustness”. In the context of hazmat transportation, when the probability threshold is very big, the robustness of the CVaR concept concerns with the worst accident location. On the other hand, the worst-case consideration in WCVaR provides protections against data inaccuracy, and its robustness concerns with the worst realizations of accident probabilities and accident consequences. Therefore, these two kinds of robustness are different in nature. Furthermore, the uncertainty of accident locations has a probability distribution, and the uncertainty of data involves intervals only.

The contributions of this paper are summarized as follows: First, we formally introduce CVaR and WCVaR in hazmat routing. Second, we provide axiomatic studies to understand the validity and meanings of the axioms proposed by Artzner et al. (1999) and Erkut and Verter (1998). Third, we devise efficient computational methods for solving CVaR and WCVaR minimization problems for hazmat routing. Fourth, we confirm the validity of CVaR and WCVaR concepts and the practicability of the proposed algorithms through a case study in a realistic road network.

We sketch the organization of this paper. In Section 2, we provide a review of hazmat routing methods reported in the literature and compare them. In Section 3, we introduce CVaR for hazmat routing and provide rigorous mathematical and computational properties as well as an axiomatic study. In Section 4, we discuss the critical issue of data uncertainty in hazmat transportation and suggest robust optimization methods. In Section 5, we introduce WCVaR along with mathematical and computational properties. In Section 6, we provide a case study in a vehicular road network in Buffalo, New York and discuss the results of numerical experiments. In Section 7, we conclude this paper.

2. Comparison of Routing Methods for Hazardous Materials

In this section, we provide a brief summary of existing approaches in hazmat routing. Let us consider a transportation network \( G = (\mathcal{N}, \mathcal{A}) \), where \( \mathcal{N} \) is the set of nodes and \( \mathcal{A} \) the set of arcs. Each arc \((i, j) \in \mathcal{A}\) is assigned two attributes: accident probability \(p_{ij}\) and accident consequence \(c_{ij}\). Values of \(c_{ij}\) can be determined by a risk assessment method, for example, the \(\lambda\)-neighborhood concept proposed by Batta and Chiu (1988), and values of \(p_{ij}\) should be collected from certain data sources.
Table 1  Mathematical Notation

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
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<tbody>
<tr>
<td>( G(\mathcal{N}, \mathcal{A}) )</td>
<td>A graph of transportation network</td>
</tr>
<tr>
<td>( \mathcal{N} )</td>
<td>Set of nodes, (</td>
</tr>
<tr>
<td>( \mathcal{A} )</td>
<td>Set of arcs, (</td>
</tr>
<tr>
<td>( p_{ij} )</td>
<td>Accident probability along arc ((i, j))</td>
</tr>
<tr>
<td>( c_{ij} )</td>
<td>Accident consequence along arc ((i, j))</td>
</tr>
<tr>
<td>( \mathcal{P} )</td>
<td>Set of all available paths for given O-D pair</td>
</tr>
<tr>
<td>( \mathcal{C} )</td>
<td>Set of ascending-order sorted arc consequences in (G)</td>
</tr>
<tr>
<td>( \mathcal{A}^l )</td>
<td>Arc set for path (l), and (</td>
</tr>
<tr>
<td>( \mathcal{C}^l )</td>
<td>Set of ascending-order sorted arc consequences for path (l)</td>
</tr>
<tr>
<td>( R^l )</td>
<td>Discrete random variable for the risk along path (l \in \mathcal{P})</td>
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</table>

All the mathematical notation used in this Section is provided in Table 1. Assume a path \(l\) consists of a set of arcs \(\mathcal{A}^l = \{(i_k, j_k) \in \mathcal{A} : k = 1, 2, \ldots, |l|\}\) where \((i_k, j_k)\) is the \(k\)-th arc in the path. Then for the computation of the risk that this path generates there exist different kinds of risk measures. The traditional method (TR) employs the expected value of the consequence along path \(l\) as a risk measure (Sherali et al. 1997, Erkut and Verter 1998):\[
E[R^l] = \sum_{(i_k, j_k) \in \mathcal{A}^l} \prod_{(i_h, j_h) \in \mathcal{A}^l, h < k} (1 - p_{i_h, j_h}) p_{i_k, j_k} c_{ij} \tag{1}
\]

An important assumption of the TR model is that if an accident occur on an arc \((i, j)\) then the transportation of the hazmat shipment terminates immediately. Using this objective, the routing problem can be formulated as a nonlinear binary integer program, which is difficult to solve.

The TR model (1) can be approximated as an additive function, which is an instance of the tractable shortest-path problem. According to statistics in North America, accidents involving hazmat are extremely low-probability events with probability ranging from \(10^{-8}\) to \(10^{-6}\) per mile traveled (Abkowitz and Cheng 1988), which indicates the following: \[
\prod_{(i_h, j_h) \in \mathcal{A}^l, h < k} (1 - p_{i_h, j_h}) \approx 1 \quad \forall k, \quad \text{or} \quad p_{ij} p_{i', j'} \approx 0 \quad \forall (i, j), (i', j') \in \mathcal{A} \tag{2}
\]

for all \(k\). Consequently, the following additive form is a good approximation of (1) (Jin and Batta 1997) : \[
E[R^l] \approx \sum_{(i, j) \in \mathcal{A}^l} p_{ij} c_{ij} \tag{3}
\]

that is much easier to optimize compared to (1), because the resultant problem is a shortest-path problem in which the product \(p_{ij} c_{ij}\) represents the cost of traversing an arc \((i, j)\). Furthermore, Erkut and Verter (1998) showed that the use of (3) results in negligible errors. Various other models (Table 2) have been developed, focusing either only on one of the two attributes or on both.

Despite its simplicity, the TR model has some drawbacks. The most worthy mentioning weakness of the model is its risk-neutral attitude that does not reflect the public feeling against hazmat...
transportation. Motivated by this flaw of the TR model, the Perceived Risk (PR) model introduces a weight parameter $q$ on the accident consequences, to appropriately reflect the public preference. Furthermore, Erkut and Ingolfsson (2000) introduced three additional models for risk-averse decision making. The Maximum Risk (MM) model measures how far the tail of the consequence distribution extends and minimizes the maximum consequence of the proposed path. The second model is called the Mean-Variance (MV) model and it is popularly used to quantify the trade-offs between return and risk of an investment portfolio. Lastly, the Disutility (DU) model uses utility theory to model the risk of transporting hazmat, providing a risk-averse attitude since the $(i+1)$-st life is valued more than the $i$-th life loss.

Bell (2007) proposes a mixed-route model considering data uncertainty; however, it is limited to accident probabilities and requires path-enumeration for application. Other works address time-varying networks (Miller-Hooks and Mahmassani 1998), equitable risk routes (Carotenuto et al. 2007), and emergency response decisions (Zografos and Androutsopoulos 2008), with consideration of traditional-risk minimization approaches. In addition, Waller and Ziliaskopoulos (2006) have applied chance-constrained optimization methods to traffic assignment problems to consider travel-demand uncertainty. While the chance constraints are equivalent to VaR constraints, our model is fundamentally different as the problem is CVaR minimization, and the main objective in hazmat problems is to reduce the risk of accidents themselves, not as constraints.

Compared with other risk measures used in hazmat transportation, we claim that CVaR is a better risk measure in the sense that it offers a control parameter—probability threshold or confidence level—that is easier to understand for flexible decision making, and it provides a risk-averse routing method. In addition, complicated CVaR problems can be solved efficiently.

<table>
<thead>
<tr>
<th>Model</th>
<th>Risk Measure</th>
<th>Objective</th>
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<th>Risk Measure</th>
<th>Objective</th>
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<tbody>
<tr>
<td>TR</td>
<td>Expected Risk</td>
<td>$\min_{l \in P} \sum_{(i,j) \in A^l} p_{ij}C_{ij}$</td>
<td>MM</td>
<td>Maximum Risk</td>
<td>$\min_{l \in P} \max_{(i,j) \in A^l} c_{ij}$</td>
</tr>
<tr>
<td>PE</td>
<td>Population Exposure</td>
<td>$\min_{l \in P} \sum_{(i,j) \in A^l} c_{ij}$</td>
<td>MV</td>
<td>Mean-Variance</td>
<td>$\min_{l \in P} \sum_{(i,j) \in A^l} (p_{ij}C_{ij} + kp_{ij}(c_{ij})^2)$</td>
</tr>
<tr>
<td>IP</td>
<td>Incident Probability</td>
<td>$\min_{l \in P} \sum_{(i,j) \in A^l} p_{ij}$</td>
<td>DU</td>
<td>Disutility</td>
<td>$\min_{l \in P} \sum_{(i,j) \in A^l} p_{ij}(\exp(kc_{ij} - 1))$</td>
</tr>
<tr>
<td>PR</td>
<td>Perceived Risk</td>
<td>$\min_{l \in P} \sum_{(i,j) \in A^l} p_{ij}(c_{ij})^q$</td>
<td>CR</td>
<td>Conditional Probability</td>
<td>$\min_{l \in P} \left( \frac{\sum_{(i,j) \in A^l} p_{ij}C_{ij}}{\sum_{(i,j) \in A^l} p_{ij}} \right)$</td>
</tr>
</tbody>
</table>
3. Conditional Value-at-Risk Minimization

Since CVaR extends VaR, we first provide a brief introduction to VaR. For a given risk probability threshold, or confidence level, $\alpha \in (0, 1)$ and a path $l \in P$, we define VaR to be the minimal level $\beta$ at which the probability that the risk $R^l$ exceeds $\beta$ is less than or equal to $1-\alpha$:

$$\text{VaR}^l_\alpha = \min\{\beta : \Pr(R^l > \beta) \leq 1-\alpha\} \quad (4)$$

Note that we use the notation introduced in Table 1. Given $P$, the corresponding VaR minimization problem is to solve

$$\text{VaR}^*_\alpha = \min_{l \in P} \text{VaR}^l_\alpha \quad (5)$$

The confidence level $\alpha$ is a single constant for the entire network; for example, $\alpha = 0.99$ for the 99-percent confidence level. Naturally, the lower the VaR value in path $l$, the more desirable the path is. That means it brings less ‘bad-case’ risk compared with other routes at the same confidence level $\alpha$. It has been shown that (5) can be solved by at most $|\mathcal{N}|$ shortest-path problems (Kang et al. 2013). Note that all the proofs of the Propositions and Lemmas are provided in the e-Companion.

**Proposition 1 (Kang et al. 2013).** For sufficiently small $\alpha > 0$, VaR becomes zero for all paths. For sufficiently large $\alpha < 1$, the VaR minimization model is equivalent to the Maximum Risk (MM) model.

With varying $\alpha$, the VaR model provides flexibility in the risk attitude from risk-indifferent to risk-averse. While risk-indifference is not desirable in hazmat routing, the CVaR model improves this weakness of the VaR model.

While CVaR minimization for financial problems has been well studied, CVaR in hazmat transportation is first formally considered in this paper. Unlike in financial problems, CVaR minimization in hazmat transportation needs to solve a mixed-integer problem because it is a network problem. For a path $l \in P$ at the confidence level $\alpha$, the CVaR for general distributions is defined as (Rockafellar and Uryasev 2002, Sarykalin et al. 2008):

$$\text{CVaR}^l_\alpha = \chi^l_\alpha \text{VaR}^l_\alpha + (1 - \chi^l_\alpha) \mathbb{E}[R^l | R^l > \text{VaR}^l_\alpha] \quad (6)$$

where $\chi^l_\alpha = (\Pr[R^l \leq \text{VaR}^l_\alpha] - \alpha)/(1 - \alpha)$. While the second component, $\mathbb{E}[R^l | R^l > \text{VaR}^l_\alpha]$, is incoherent (Acerbi 2002, 2004), the entire $\text{CVaR}^l_\alpha$ in (6) is a coherent risk measure (Rockafellar and Uryasev 2002, Pflug 2000), and the expectation in the hazmat context can be approximated by:

$$\mathbb{E}[R^l | R^l > \text{VaR}^l_\alpha] \approx \sum_{(i,j) \in A^l : c_{ij} > \text{VaR}^l_\alpha} p_{ij} c_{ij} \quad (7)$$
where the risk $R^l$ along the path $l$ has the following discrete distribution:

$$\Pr\{R^l = x\} = \begin{cases} 
1 - \sum_{(i,j) \in A^l} (1 - p_{i_h,j_h}) p_{i_h,j_k} & \text{if } x = 0 \\
\prod_{(i,j) \in A^l, h < k} (1 - p_{i_h,j_h}) p_{i_h,j_k} & \text{if } x = c_{i_k,j_k} \forall (i_k, j_k) \in A^l
\end{cases} \quad (8)$$

with indexes $k$ and $h$ being integers such that $1 \leq k \leq |A^l|, 1 \leq h \leq |A^l|$.

The distribution (8) is approximated by

$$\Pr\{R^l = x\} \approx \begin{cases} 
1 - \sum_{(i,j) \in A^l} p_{ij} & \text{if } x = 0 \\
p_{ij} & \text{if } x = c_{ij} \forall (i, j) \in A^l
\end{cases} \quad (9)$$

It is unclear how to consider the measure $\text{CVaR}^l_\alpha$ given in the form (6) with an approximation (7) as an objective function for the CVaR minimization problem, because of the conditioning in the expectation.

Let us consider the following auxiliary function (Rockafellar and Uryasev 2000, Pflug 2000) and its approximation:

$$\Phi^l_\alpha(r) = r + \frac{1}{1 - \alpha} \mathbb{E}[R^l - r]^+ \approx r + \frac{1}{1 - \alpha} \left\{ \left( 1 - \sum_{(i,j) \in A^l} p_{ij} \right) [0 - r]^+ + \sum_{(i,j) \in A^l} p_{ij} [c_{ij} - r]^+ \right\} \quad (10)$$

where we denote $[x]^+ = \max(x, 0)$. Then we can show that the CVaR for path $l$ can be measured by

$$\text{CVaR}^l_\alpha = \min_{r \in \mathbb{R}} \Phi^l_\alpha(r) \approx \min_{r \in \mathbb{R}} \left[ r + \frac{1}{1 - \alpha} \left\{ \left( 1 - \sum_{(i,j) \in A^l} p_{ij} \right) [0 - r]^+ + \sum_{(i,j) \in A^l} p_{ij} [c_{ij} - r]^+ \right\} \right] \quad (11)$$

$$= \min_{r \in \mathbb{R}^+} \left[ r + \frac{1}{1 - \alpha} \sum_{(i,j) \in A^l} p_{ij} [c_{ij} - r]^+ \right] \quad (12)$$

Note that in (12) we enforced $r$ being nonnegative. The equivalence of (11) and (12) can be easily checked by comparing the cases when $r$ is a negative number and when $r = 0$ for all $\alpha \in (0, 1)$. We will use the form (12) in the most parts of this paper.

Finally the CVaR minimization is equivalent to minimizing $\Phi^l_\alpha$ by choosing a path $l \in \mathcal{P}$ at the confidence level $\alpha$. That is,

$$\min_{l \in \mathcal{P}} \text{CVaR}^l_\alpha = \min_{l \in \mathcal{P}, r \in \mathbb{R}^+} \Phi^l_\alpha(r) \approx \min_{l \in \mathcal{P}, r \in \mathbb{R}^+} \left[ r + \frac{1}{1 - \alpha} \sum_{(i,j) \in A^l} p_{ij} [c_{ij} - r]^+ \right] \quad (13)$$

### 3.1. Axiomatic Study for CVaR

The CVaR is known to satisfy the four axioms by Artzner et al. (1999) to be a coherent risk measure for general loss distributions (Rockafellar and Uryasev 2002). The four axioms for a risk measure $\xi$ that maps a random loss $X$ to a real number are:
CA1 (Translation Invariance) For any real number $m$, $\xi(X + m) = \xi(X) + m$.

CA2 (Subadditivity) For all $X_1$ and $X_2$, $\xi(X_1 + X_2) \leq \xi(X_1) + \xi(X_2)$.

CA3 (Positive Homogeneity) For all $\lambda \geq 0$, $\xi(\lambda X) = \lambda \xi(X)$.

CA4 (Monotonicity) For all $X_1$ and $X_2$ with $X_1 \leq X_2$ a.s., $\xi(X_1) \leq \xi(X_2)$.

The question is if the approximated version of CVaR in (11) is still coherent. Note that the approximated distribution (9) constitutes a complete probability distribution, and the approximated CVaR in (11) is the exact CVaR for the distribution (9). Since CVaR is coherent for the distribution (9), the approximated CVaR measure in (11) is also coherent. We state this finding as a proposition:

**Proposition 2.** The approximated CVaR in (11) for hazmat routing is a coherent risk measure in the sense of Artzner et al. (1999).

While CVaR in hazmat transportation—both exact and approximated—is a coherent risk measure, we find that the translation invariance axiom CA1 is less meaningful and the monotonicity axiom CA4 is not applicable in the context of hazmat transportation. The meaning of $X + m$ in CA1 in hazmat transportation is that the accident consequence increases by $m$, even when there is no accident. Although CA1 is mathematically correct and valid, it reads improperly in the context of hazmat transportation. We also note that the condition in the monotonicity axiom CA4 is not applicable to hazmat routing, as the axiom considers the case when the relationship between two random variables is that $R_{l1} \leq R_{l2}$ a.s., i.e. $\Pr[R_{l1} > R_{l2}] = 0$. In hazmat routing, $\Pr[R_{l1} > R_{l2}]$ is never zero, because it is possible that $R_{l2} = 0$ when $R_{l1} > 0$ as long as $p_{ij} > 0$ for some $(i, j) \in A_{l1}$.

We can show that the approximated CVaR in hazmat routing is a coherent risk measure.

While the meaning of the positive homogeneity axiom CA3 is clear in hazmat transportation, the meaning of the subadditivity axiom CA2 is ambiguous. We provide a “translation” of CA2 in the context of hazmat transportation with the approximate CVaR measure (11). Artzner et al. (1999) states the meaning of CA2 as “a merger does not create extra risk”. In the context of hazmat, we may similarly state as follows: using two paths by a same carrier does not create extra risk. We consider two such cases. First, when two different trucks from the same carrier use two different paths, say $l_1$ and $l_2$. Then $X_1 + X_2$ in CA2 requires a joint probability of $R_{l1}^1 + R_{l2}^2$ where each is of the form (8). However, with the approximation (2), we can write:

$$
\Pr\{R_{l1}^1 + R_{l2}^2 = z\} \approx \begin{cases} 
1 - \sum_{(i,j)\in A_{l1} \cup A_{l2}} p_{ij} & \text{if } z = 0 \\
\frac{1}{p_{ij}} & \text{if } z = c_{ij} \quad \forall (i,j) \in A_{l1} \cup A_{l2}
\end{cases}
$$

Note that $\Pr\{R_{l1}^1 + R_{l2}^2 = c_{ij} + c_{i'j'}\} \approx 0$ for any $(i, j) \in A_{l1}$ and $(i', j') \in A_{l2}$, which means that the probability of one accident along path $l_1$ and another accident along path $l_2$ at the same time is very small. Alternatively, we can consider a truck traveling both paths $l_1$ and $l_2$ within one trip,
when $l_2$ starts where $l_1$ ends. In both cases, the translations of CA2 are same with approximation (2). The translation follows.

CA2' (Subadditivity) For all paths $l_1$ and $l_2$,

$$\min_{r \in \mathbb{R}^+} \left[ r + \frac{1}{1-\alpha} \left\{ \sum_{(i,j) \in A_1^l} p_{ij} [c_{ij} - r]^+ + \sum_{(i,j) \in A_2^l} p_{ij} [c_{ij} - r]^+ \right\} \right] \leq \min_{s \in \mathbb{R}^+} \left[ s + \frac{1}{1-\alpha} \sum_{(i,j) \in A_1^l} p_{ij} [c_{ij} - s]^+ \right] + \min_{t \in \mathbb{R}^+} \left[ t + \frac{1}{1-\alpha} \sum_{(i,j) \in A_2^l} p_{ij} [c_{ij} - t]^+ \right].$$

Erkut and Verter (1998) proposed three axioms that path evaluation models in hazmat routing should satisfy. Let us denote a risk measure of path $l$ by $\zeta_l(p, c)$, where $p$ is a vector of accident probabilities and $c$ is a vector of accident consequences.

HA1 (Monotonicity) If path $l_1$ is contained in path $l_2$, then $\zeta_{l_1} \leq \zeta_{l_2}$.

HA2 (Optimality Principle) If $\zeta_{l_2} = \min_{l \in P_2} \zeta_l$ and path $l_1$ is contained in path $l_2$, then $\zeta_{l_1} = \min_{l \in P_1} \zeta_l$, where $P_1$ and $P_2$ are the sets of all paths connecting the origin and the destination of path $l_1$ and path $l_2$, respectively.

HA3 (Attribute Monotonicity) For any vectors $m \geq 0$ and $n \geq 0$, $\zeta_l(p, c) \leq \zeta_l(p + m, c + n)$ for all path $l \in P$.

Note that the Traditional Risk model violates all three axioms, unless approximation (2) is used; with approximation (2), the Traditional Risk model satisfies all three axioms (Erkut and Verter 1998, Erkut and Ingolfsson 2005). There are many other models that satisfy all three axioms with the approximation. There are some models that satisfy HA1 and HA3, but violate HA2 (Erkut and Verter 1998). We can show that the CVaR model with approximation as in (11) is one of such models.

**Proposition 3.** The CVaR measure with approximation (11) satisfies Axioms HA1 and HA3 for path evaluation models in hazmat routing.

Figure 1 illustrates an example when CVaR violates HA2. For traveling from node 1 to node 4, there are two paths $l_1$ and $l_2$, and path $l_1$ is optimal with respect to CVaR minimization. However, for traveling from node 2 to node 3, path $l_3$, which is contained in path $l_1$, is not optimal; path $l_4$ is optimal. Since the CVaR model violates the path-evaluation optimality principle, we could not use a labeling algorithm to solve the model. In the subsequent section, we propose a computational method that finds an exact optimal solution by solving a finite number of shortest-path problems.

### 3.2. Computational Scheme for CVaR Minimization

The CVaR minimization problem can be written as

$$\min_{r \in \mathbb{R}^+} \left( r + \frac{1}{1-\alpha} \min_{x \in \Omega} \sum_{(i,j) \in A} p_{ij} [c_{ij} - r]^+ x_{ij} \right)$$

(15)
Figure 1 An example showing that CVaR violates the path-selection optimality principle suggested by Erkut and Verter (1998).

where

$$\Omega \equiv \left\{ x : \sum_{(i,j) \in A} x_{ij} - \sum_{(j,i) \in A} x_{ji} = b_i \quad \forall i \in \mathcal{N}, \text{ and } x_{ij} \in \{0,1\} \quad \forall (i,j) \in \mathcal{A} \right\} \quad (16)$$

and the parameter $b_i$ has the following values:

$$b_i = \begin{cases} 1 & \text{if node } i \text{ is the source} \\ -1 & \text{if node } i \text{ is the sink} \\ 0 & \text{otherwise} \end{cases}$$

We first observe that an optimal solution $r^*$ to the problem (15) is bounded by the largest accident consequence among all arcs, i.e., $r^* \leq \max_{(i,j) \in \mathcal{A}} c_{ij}$. For $r > \max_{(i,j) \in \mathcal{A}} c_{ij}$, $[c_{ij} - r]^+ = 0$ for all $(i,j) \in \mathcal{A}$. We obtain the following result.

**Proposition 4.** There exists an optimal solution, $r^*$, to the CVaR minimization problem (15) in the following set:

$$r^* \in \{0\} \cup \{c_{ij} : (i,j) \in \mathcal{A}\} \quad (17)$$

While examining the set $\{0\} \cup \{c_{ij} : (i,j) \in \mathcal{A}\}$, we shall keep a record of the minimum CVaR value found so far,

$$w^t = \hat{c}_k + \frac{1}{1 - \alpha} \min_{x \in \Omega} \sum_{(i,j) \in \mathcal{A}} p_{ij}[c_{ij} - \hat{c}_k]^+ x_{ij}$$

where $\hat{c}_k$ is the $k$-th element of $\{0\} \cup \{c_{ij} : (i,j) \in \mathcal{A}\}$ that we examine. Then, if the next value that we examine, $r_n$, is already greater than the current $w^t$, we do not need to solve the shortest-path sub-problem since it always gives a nonnegative cost value.

We propose the following computational scheme:

**A Computational Scheme for the CVaR minimization problem (13)**

**Step 0.** Let $\hat{C}$ be an ordered set of $\{0\} \cup \{c_{ij} : (i,j) \in \mathcal{A}\}$ and $\hat{c}_k$ denote $k$-th smallest element of $\hat{C}$, where $\hat{c}_0 = 0$. Set $w^t \leftarrow \infty$ and $k \leftarrow 0$.

**Step 1.** If $\hat{c}_k \geq w^t$, go to Step 3.
Step 2. For $\hat{c}_k$, obtain the following value:

$$w^k = \hat{c}_k + \frac{1}{1 - \alpha} \min_{x \in \Omega} \sum_{(i,j) \in A} p_{ij} [c_{ij} - \hat{c}_k]^+ x_{ij}$$

by solving the corresponding shortest-path sub-problem. If $w^k < w^\#$, set $w^\# \leftarrow w^k$.

Step 3. Update $k \leftarrow k + 1$ and repeat Steps 1 and 2, until $k = |A|$.

3.3. Properties of CVaR Minimization

In this section, we show some properties of the CVaR model in hazmat transportation, compared with other existing models such as the Traditional Risk (TR) and Maximum Risk (MR) models. Most importantly, we show that the CVaR minimization delivers a risk-neutral solution as in the traditional risk model, even to the least risk-averse decision makers ($\alpha \to 0$). We note that VaR minimization may deliver a risk-indifferent solution, as noted in Proposition 1.

We have the following results for CVaR minimization:

**Proposition 5.** There exists a scalar $\alpha_{\text{min}}$, such that $l^*_{\text{CVaR}} = l^*_{\text{TR}}$, $\forall \alpha \in (0, \alpha_{\text{min}})$, where $l^*_{\text{CVaR}}$ and $l^*_{\text{TR}}$ are the optimal paths determined by the CVaR and TR models respectively. In particular,

$$\alpha_{\text{min}} = \min_{l \in P} \left( 1 - \sum_{(i,j) \in A^l} p_{ij} \right)$$

**Proposition 6.** There exists a scalar $\alpha_{\text{max}}$, such that $l^*_{\text{CVaR}} = l^*_{\text{MM}}$, $\forall \alpha \in (\alpha_{\text{max}}, 1)$, where $l^*_{\text{CVaR}}$ and $l^*_{\text{MM}}$ are the optimal paths determined by the CVaR and MM models respectively. In particular,

$$\alpha_{\text{max}} = \max_{l \in P} p_{\text{max}}^l$$

where $p_{\text{max}}^l = \Pr[R^l = \max_{(i,j) \in A^l} c_{ij}]$ for all $l \in P$, which is the accident probability of the arc with the greatest accident consequence in a path $l$.

In Propositions 5 and 6, we observe that the CVaR model offers a flexible tool for decision makers whose risk attitudes are risk-neutral (TR model) to risk-averse (MM model). This is an improvement over the VaR model. As $\alpha_{\text{min}}$ can be as big as 0.9999 in a realistic hazmat application, for which VaR becomes zero and CVaR becomes TR, it is easy to obtain a point-less result with the VaR model if it is used without caution; however, with the CVaR model, we still obtain the least TR path.

4. Data Uncertainty and Robust Routing

There are two kinds of uncertain factors in hazmat transportation problems. When we want to minimize the expected accident consequence, we first need to identify the probability distribution
of the loss. Such identification is related to this question: In which road segment and at what probability will an accident occur? This is the first uncertain factor.

The second uncertainty comes from the data \( p_{ij} \) and \( c_{ij} \), which constitute the distribution of loss. If the data \( p_{ij} \) and \( c_{ij} \) are provided, we can determine a safe route using various methods. However, for hazmat problems, accurate estimates of the data and their distribution information are rarely available, because hazmat accidents are extremely low-probability events. In addition to this property of hazmat accidents, the accident consequences depend on the weather conditions (Akgun et al. 2007) at the time of the accident as well as on the nature of accidents and hazmat types. Therefore, accident consequences are hard to estimate.

Due to the insufficiency of available data, any stochastic-programming-based approach becomes impractical. What we can obtain at best would be interval data of the accident probabilities and consequences. That is, we may be able to obtain the minimum possible and maximum possible values, without distributional information. In such cases, robust optimization methods considering the worst case are most appropriate. For financial portfolio optimization, worst-case CVaR optimization methods have been proposed (Zhu and Fukushima 2009, Čerňáková 2006), but the hazmat routing application has completely different challenges of computability.

To explain the differences, let us denote the uncertain probabilities and consequences by \( \tilde{p}_{ij} \) and \( \tilde{c}_{ij} \). When interval data are available, we have \( \tilde{p}_{ij} \in [p_{ij}, p_{ij} + q_{ij}] \) and \( \tilde{c}_{ij} \in [c_{ij}, c_{ij} + d_{ij}] \), where \( q_{ij} \) and \( d_{ij} \) are positive constants. For more general cases, we can write \( \tilde{p} \in \mathcal{U}_p \) and \( \tilde{q} \in \mathcal{V}_c \). We call \( \mathcal{U}_p \) and \( \mathcal{V}_c \) the set of uncertainty. The uncertain formulation of the Traditional Risk model that minimizes the expected risk is

\[
\min_{x \in \Omega} \max_{\tilde{p} \in \mathcal{U}_p, \tilde{q} \in \mathcal{V}_c} \sum_{(i,j) \in A} \tilde{p}_{ij} \tilde{c}_{ij} x_{ij} \tag{18}
\]

which is a robust shortest-path problem considering the worst-case scenario.

While a few robust shortest-path problems have already been solved (Kouvelis and Yu 1996, Bertsimas and Sim 2003, Chaerani et al. 2005), we face a new class of robust shortest-path problems with two uncertain multiplicative cost coefficients. In the already solved problems, the cost coefficients are considered as a single uncertain-cost vector, such as in \( \tilde{c}^T x \). On the other hand, in problem (18), we have two uncertain-cost parameters, \( \tilde{p}_{ij} \) and \( \tilde{c}_{ij} \), of different characteristics. Data sources for two parameters are different, and the nature of uncertainty is also different; therefore, the two uncertain parameters cannot be regarded as a single-cost parameter in a robust optimization framework. Hence, we need a new method to solve the robust shortest-path problem in (18). It is also unclear if these two parameters are correlated or not (Kwon et al. 2013).

In the next section, we extend the method of Kwon et al. (2013) to solve the worst-case CVaR minimization problems.
5. Worst-case Conditional Value-at-Risk

The worst-case CVaR (WCVaR) minimization can be studied in extension of the robust shortest-path problem (18). When data are uncertain, we define the following WCVaR measure:

\[
\text{WCVaR}^l_\alpha(U_p, V_c) = \max_{\tilde{p} \in U_p, \tilde{c} \in V_c} \min_{r \in \mathbb{R}} \phi^l_\alpha(r; \tilde{p}, \tilde{c}) = \max_{\tilde{p} \in U_p, \tilde{c} \in V_c} \min_{r \in \mathbb{R}} \left\{ r + \left(1 - \sum_{(i,j) \in A^l} \tilde{p}_{ij} \right)[0-r]^+ + \frac{1}{1-\alpha} \sum_{(i,j) \in A^l} \tilde{c}_{ij} - r \right\} = \max_{r \in \mathbb{R}^+} \min_{\tilde{p} \in U_p, \tilde{c} \in V_c} \left\{ r + \frac{1}{1-\alpha} \sum_{(i,j) \in A^l} \tilde{p}_{ij} \tilde{c}_{ij} - r \right\} = \min_{r \in \mathbb{R}^+} \max_{\tilde{p} \in U_p, \tilde{c} \in V_c} \left\{ r + \frac{1}{1-\alpha} \sum_{(i,j) \in A^l} \tilde{p}_{ij} \tilde{c}_{ij} - r \right\}
\]

where \(\tilde{p}\) and \(\tilde{c}\) are the uncertain parameters, and \(U\) and \(V\) are compact and convex sets. The switch of ‘\(\min\)’ and ‘\(\max\)’ in (19) is valid by Theorem 2 of Zhu and Fukushima (2009).

5.1. Axiomatic Study of WCVaR

Since the WCVaR measure in (19) is the worst-case CVaR measure for an uncertain (or ambiguous) probability distribution, the coherence of WCVaR in (19) can be shown directly by the following proposition by Zhu and Fukushima (2009) and Proposition 2:

**Proposition 7** (Zhu and Fukushima 2009). If \(\rho\) associated with crisp (or determinate) probability measure \(P\) is a coherent risk measure, then the corresponding \(\rho_w \equiv \sup_{P \in \mathcal{P}} \rho(X)\) associated with ambiguous probability measure \(\mathcal{P}\) remains a coherent risk measure.

In addition, it is natural that the WCVaR measure satisfies Axioms HA1 and HA3. However, we need to redefine Axiom HA3 so that it is meaningful with data-uncertainty and WCVaR.

(AHa3') (Attribute Monotonicity) For any vectors \(m \geq 0\) and \(n \geq 0\), \(\text{WCVaR}^l_\alpha(U_p, V_c) \leq \text{WCVaR}^l_\alpha(U_p + m, V_c + n)\) for all paths \(l \in \mathcal{P}\) and all \(\alpha \in (0, 1)\), where

\[
U_p + m = \{ p + m : p \in U_p \}
\]
\[
V_c + n = \{ c + n : c \in V_c \}
\]

We can also consider the following alternative:

(AHa3") (Attribute Monotonicity) For any compact and convex sets \(U_p, V_c, U_p'\) and \(V_c'\), \(\text{WCVaR}^l_\alpha(U_p, V_c) \leq \text{WCVaR}^l_\alpha(U_p', V_c')\) for all path \(l \in \mathcal{P}\) and all \(\alpha \in (0, 1)\), where \(U_p \preceq U_p'\) and \(V_c \preceq V_c'\) meaning that

\[
p \leq p' \quad \forall p \in U_p, \; p' \in U_p' \quad \text{or, sup } U_p \leq \inf U_p'
\]
\[
c \leq c' \quad \forall c \in V_c, \; c' \in V_c' \quad \text{or, sup } V_c \leq \inf V_c'
\]
Proposition 8. When $\mathcal{U}$ and $\mathcal{V}$ are compact and convex sets, the worst-case CVaR (WCVaR) measure

$$\text{WCVaR}_l(\mathcal{U}_p, \mathcal{V}_c) = \min_{r \in \mathbb{R}^+} \max_{\tilde{p} \in \mathcal{U}_p, \tilde{c} \in \mathcal{V}_c} \left\{ r + \frac{1}{1 - \alpha} \sum_{(i,j) \in A} \tilde{p}_{ij} [\tilde{c}_{ij} - r]^+ \right\}$$

is a coherent risk measure and satisfies Axioms HA1 and HA3 (both HA3' and HA3")..

The WCVaR minimization problem applied to hazmat transportation is as follows:

$$\min_{l \in \mathcal{P}} \text{WCVaR}_l(\mathcal{U}_p, \mathcal{V}_c) = \min_{l \in \mathcal{P}} \min_{r \in \mathbb{R}^+} \max_{\tilde{p} \in \mathcal{U}_p, \tilde{c} \in \mathcal{V}_c} \left( r + \frac{1}{1 - \alpha} \sum_{(i,j) \in A} \tilde{p}_{ij} [\tilde{c}_{ij} - r]^+ \right)$$

$$= \min_{r \in \mathbb{R}^+} \left( r + \frac{1}{1 - \alpha} \min_{x \in \Omega} \max_{\tilde{p} \in \mathcal{U}_p, \tilde{c} \in \mathcal{V}_c} \sum_{(i,j) \in A} \tilde{p}_{ij} [\tilde{c}_{ij} - r]^+ x_{ij} \right)$$

where the set $\Omega$ is defined in (16). For each $r$, the WCVaR minimization problem requires us to solve a robust shortest-path problem with two uncertain multiplicative cost coefficients. We will extend the solution approach proposed by Kwon et al. (2013) for solving the sub robust shortest-path problem.

5.2. Box-Constrained Uncertainty Set

Let us consider the uncertain parameters $\tilde{p}$ and $\tilde{c}$ given in the following budgeted box-constrained uncertainty set, as considered by Kwon et al. (2013).

$$\tilde{p}_{ij} = p_{ij} + q_{ij} u_{ij}$$

$$\tilde{c}_{ij} = c_{ij} + d_{ij} v_{ij}$$

where $q_{ij}$ and $d_{ij}$ are given constants,

$$u_{ij} \in \mathcal{U} = \left\{ u : 0 \leq u_{ij} \leq 1 \quad \forall (i,j), \quad \sum_{(i,j) \in A} u_{ij} \leq \Gamma_u \right\}$$

$$v_{ij} \in \mathcal{V} = \left\{ v : 0 \leq v_{ij} \leq 1 \quad \forall (i,j), \quad \sum_{(i,j) \in A} v_{ij} \leq \Gamma_v \right\}$$

and $\Gamma_u$ and $\Gamma_v$ are positive integers. An intuitive description of these parameters is given in Section 6.1. The parameters $\Gamma_u$ and $\Gamma_v$ are called the budgets of uncertainty, and represent the level of ambiguity in the data. Increasing the values of $\Gamma_u$ and $\Gamma_v$ increases the level of robustness in the objective (Bertsimas and Sim 2003). Replacing the uncertain parameters $\tilde{p}$ and $\tilde{c}$ in (20) with the corresponding budgeted box-constrained set, we obtain:

$$\min_{r \in \mathbb{R}^+} \left( r + \frac{1}{1 - \alpha} \min_{x \in \Omega} \max_{u \in \mathcal{U}, v \in \mathcal{V}} \sum_{(i,j) \in A} (p_{ij} + q_{ij} u_{ij}) [(c_{ij} + d_{ij} v_{ij}) - r]^+ x_{ij} \right)$$

We first observe that $v$ at optimum is binary.
Lemma 1. There always exists a binary \( v \) that is a solution to the WCVaR minimization problem (26). In particular, for any given \( r, x, \) and \( u, \) the maximization problem of \( v \)

\[
\max_{v \in V} \sum_{(i,j) \in A} (p_{ij} + q_{ij} u_{ij}) \left( [(c_{ij} + d_{ij}) v_{ij}] - r \right) x_{ij}
\]

has a binary solution.

Note that \( [(c_{ij} + d_{ij}) v_{ij}] - r \) takes values in the interval \( [c_{ij} - r, c_{ij} + d_{ij} - r] \). Let us define

\[
e_{ij}(r) = [c_{ij} - r]
\]

\[
f_{ij}(r) = [c_{ij} + d_{ij} - r] - [c_{ij} - r]
\]

A linearizing transformation from \( [(c_{ij} + d_{ij}) v_{ij}] - r \) to \( e_{ij}(r) + f_{ij}(r) v_{ij} \) changes the shape of the objective function; the former is nonlinear and the latter is linear in \( v_{ij} \). However, when they are maximized with respect to \( v \) over the set \( V \), they have same binary solutions by Lemma 1. Therefore, we can rewrite the inner min-max problem in (26) equivalently as follows:

\[
\min_{x \in \Omega} \max_{u \in U, v \in V} \sum_{(i,j) \in A} (p_{ij} + q_{ij} u_{ij}) \left( [(c_{ij} + d_{ij}) v_{ij}] - r \right) x_{ij}
\]

\[
= \min_{x \in \Omega} \max_{u \in U, v \in V} \sum_{(i,j) \in A} (p_{ij} + q_{ij} u_{ij}) (e_{ij}(r) + f_{ij}(r) v_{ij}) x_{ij} \equiv \text{RSP}(r)
\]

We observe that the inner maximization problem in (28) is a non-convex disjoint bilinear program for any given \( x \) and \( r \), for which an optimal solution exists at an extreme point (Floudas and Visweswaran 1994). Therefore we can obtain binary optimal solutions \( u \) and \( v \). We obtain the following result.

Proposition 9. There exists an optimal solution, \( r^* \), to the WCVaR minimization problem (26) in the following set:

\[
r^* \in \mathcal{R} \equiv \{ 0 \} \cup \{ c_{ij} : (i, j) \in A \} \cup \{ c_{ij} + d_{ij} : (i, j) \in A \}
\]

Proposition 9 indicates that for each value of \( r \) in the set \( \mathcal{R} \), we need to solve a corresponding robust shortest-path problem \( \text{RSP}(r) \) to obtain a solution to the WCVaR minimization problem.

5.3. Solving the Robust Shortest Path Problems

We briefly illustrate the dual variable enumeration method proposed by Kwon et al. (2013) for solving robust shortest path problems with two multiplicative uncertain coefficients such as the inner maximization problem in \( \text{RSP}(r) \):

\[
\text{RSP}(r) = \min_{x \in \Omega} \left[ \sum_{(i,j) \in A} p_{ij} e_{ij}(r) x_{ij} + \max_{u \in U, v \in V} \sum_{(i,j) \in A} \left( q_{ij} e_{ij}(r) x_{ij} u_{ij} + p_{ij} f_{ij}(r) x_{ij} v_{ij} + q_{ij} f_{ij}(r) x_{ij} u_{ij} v_{ij} \right) \right]
\]

(30)
First, we linearize the inner maximization problem in (30) as follows: for any 
\[ \max_{u, v, w} \sum_{(i, j) \in A} \left( q_{ij} e_{ij}(r)x_{ij}u_{ij} + p_{ij} f_{ij}(r)x_{ij}v_{ij} + q_{ij} f_{ij}(r)x_{ij}w_{ij} \right) \]  
subject to
\[ u_{ij} \leq 1 \quad (\rho_{ij}) \]
\[ v_{ij} \leq 1 \quad (\mu_{ij}) \]
\[ -u_{ij} + w_{ij} \leq 0 \quad (\eta_{ij}) \]
\[ -v_{ij} + w_{ij} \leq 0 \quad (\pi_{ij}) \]
\[ \sum_{(i, j) \in A} u_{ij} \leq \Gamma_u \quad (\theta_u) \]
\[ \sum_{(i, j) \in A} v_{ij} \leq \Gamma_v \quad (\theta_v) \]
\[ u_{ij}, v_{ij}, w_{ij} \geq 0 \]

When \( \Gamma_u \) and \( \Gamma_v \) are positive integers, we can easily show that the polytope defined by the constraints of problem (31) is integral; hence, the optimal solution of problem (31) is integral.

Now consider the dual problem of problem (31) with the corresponding dual variables in parentheses:
\[ \min_{\theta_u, \theta_v, \rho, \mu, \eta, \pi} \left[ \Gamma_u \theta_u + \Gamma_v \theta_v + \sum_{(i, j) \in A} (\rho_{ij} + \mu_{ij}) \right] \]  
subject to
\[ \rho_{ij} - \eta_{ij} + \theta_u \geq q_{ij} e_{ij}(r)x_{ij} \]  
\[ \mu_{ij} - \pi_{ij} + \theta_v \geq p_{ij} f_{ij}(r)x_{ij} \]  
\[ \eta_{ij} + \pi_{ij} \geq q_{ij} f_{ij}(r)x_{ij} \]  
\[ \rho_{ij}, \mu_{ij}, \eta_{ij}, \pi_{ij}, \theta_u, \theta_v \geq 0 \]

Therefore RSP(\( r \)) can be written as follows:
\[ \text{RSP}(r) = \min_{x, \theta_u, \theta_v, \rho, \mu, \eta, \pi} \left[ \Gamma_u \theta_u + \Gamma_v \theta_v + \sum_{(i, j) \in A} (p_{ij} e_{ij}(r)x_{ij} + \rho_{ij} + \mu_{ij}) \right] \]  
subject to
\[ x \in \Omega, (33), (34), (35), \text{ and } (36) \]

The above problem (37) is a mixed integer linear programming (MILP) problem for any given \( r \). We may solve the problem (37) by a commercial optimization solver such as CPLEX, or by computational methods proposed by Kwon et al. (2013) that require solutions of finite number of (deterministic) shortest-path problems. We summarized a method of Kwon et al. (2013) in the e-Companion.
5.4. Computational Scheme for WCVaR Minimization

In support of Proposition 9, the WCVaR minimization problem (26) can be written as follows:

$$WCVaR^*_\alpha = \min_{r \in R} RSP(r)$$

where $R \equiv \{0\} \cup \{c_{ij} : (i, j) \in A\} \cup \{c_{ij} + d_{ij} : (i, j) \in A\}$ as defined in (29). In an actual implementation of the solution procedure for WCVaR, we can reduce the number of $r$ values to be considered. Since $RSP(r)$ always has nonnegative value, we do not need to consider $r$ values greater than the minimum value found so far, as similar as in the CVaR case. We propose the following computational scheme:

**A Computational Scheme for the WCVaR minimization problem (26)**

**Step 0.** Let $R$ be an ordered set of $\{0\} \cup \{c_{ij} : (i, j) \in A\} \cup \{c_{ij} + d_{ij} : (i, j) \in A\}$ and $r_k$ denote $k$-th smallest element of $R$ where $r_0 = 0$. Set $w^k \leftarrow \infty$ and $k \leftarrow 0$.

**Step 1.** If $r_k \geq w^k$, terminate the algorithm.

**Step 2.** For $r_k$, solve the following robust shortest path problem:

$$RSP(r_k) = \min_{x \in \Omega} \max_{u \in U, v \in V} \sum_{(i,j) \in A} (p_{ij} + q_{ij} u_{ij})(e_{ij}(r_k) + f_{ij}(r_k)v_{ij})x_{ij}$$

using an MILP solver for the dual form (37) or a method presented in Kwon et al. (2013) (a brief summary in the e-Companion) where

$$e_{ij}(r_k) = [c_{ij} - r_k]^+$$

$$f_{ij}(r_k) = [c_{ij} + d_{ij} - r_k]^+ - [c_{ij} - r_k]^+$$

Then compute

$$w^k = r_k + \frac{1}{1 - \alpha} RSP(r_k)$$

If $w^k < w^k$, set $w^k \leftarrow w^k$.

**Step 3.** Update $k \leftarrow k + 1$ and repeat Steps 1 and 2, until $k = 2|A| + 1$.

At termination, we find the WCVaR value as $w^k$, and the optimal path as the path obtained by the corresponding robust shortest-path problem.

5.5. Properties of WCVaR Minimization

Similar to the CVaR properties, we obtain the following results:

**Proposition 10.** For sufficiently small $\alpha > 0$, the WCVaR minimization is equivalent to the worst-case Traditional Risk (TR) model. That is

$$\min_{l \in \mathcal{P}} WCVaR^l_\alpha = \min_{l \in \mathcal{P}} \max_{\tilde{p} \in \mathcal{U}_P, \tilde{e} \in \mathcal{V}_C} \sum_{(i,j) \in A^l} \tilde{p}_{ij} \tilde{e}_{ij}$$
Proposition 11. For sufficiently large $\alpha < 1$, the WCVaR minimization model is equivalent to the worst-case Maximum Risk (MM) model. That is

$$\min_{l \in P} \text{WCVaR}^{\alpha}_l = \min_{l \in P} \max_{i \in \mathcal{V}} \max_{(i,j) \in \mathcal{A}} \tilde{c}_{ij} = \min_{l \in P} \max_{(i,j) \in \mathcal{A}} \left(c_{ij} + d_{ij}\right)$$

6. Numerical Experiments
Through numerical experiments in a realistic road network, we demonstrate the characteristics of least-CVaR and least-WCVaR paths and confirm the analytical findings.

6.1. Test Network and Data Analysis
The proposed models were tested in a portion of an actual vehicular road network in Buffalo, New York, USA, as shown in Figure 2. The understudy network consists of 90 nodes and 149 arcs, a unique origin-destination pair (OD pair), and a single hazmat shipment that needs to be transported from the origin to the destination. The population data were obtained from the U.S. Census Bureau (US Census Bureau 2010).
For every arc we need to specify two attributes: accident probabilities \( p_{ij} \) and accident consequences \( c_{ij} \). To obtain the nominal accident probabilities, we used the following formula:

\[
p_{ij} = 3.16622 \times 10^{-7} \times \text{(length of arc } (i,j))
\]  

(38)

where \( 3.16622 \times 10^{-7} \) is the hazmat accident rate per mile/vehicle (Federal Motor Carrier Safety Administration 2001). Hence, \( p_{ij} \) can be interpreted as the expected accident probability on arc \((i,j)\).

For the computation of the accident consequences \( c_{ij} \), we considered the population density in the hazmat impact zone. Specifically, we estimated the population density in a circle of radius \( \lambda \), which is equal to the hazmat spread radius as shown in Figure 3, that is commonly considered in the literature (Erkut and Verter 1998, Erkut et al. 2007). The formula used for the computation of the accident consequences was the following:

\[
c_{ij} = \pi \cdot \lambda^2 \cdot \rho_{ij}
\]  

(39)

where \( \rho_{ij} \) is the average population density along arc \((i,j)\) and \( \lambda \) is assumed to be equal to 1 mile.

For the worst-case models, we specified the values for the worst-case deviations of accident probability and accident consequence: \( q_{ij} \) and \( d_{ij} \), respectively. The following worst-case scenarios were assumed:

1. Accident probabilities can at most be doubled, i.e. \( q_{ij} = p_{ij} \).

2. Hazmat spread radius can be as far as 1.5 miles. That is, an increase of 0.5 mile in the radius \( \lambda \) of the endangered area as shown in the shaded area in Figure 3; therefore, \( d_{ij} = \pi \cdot (1.5)^2 \cdot \rho_{ij} - c_{ij} \).

Additionally, we set the values for \( \Gamma_u \) and \( \Gamma_v \), which represent the ambiguity level of data or the budget of uncertainty, as \( \Gamma_u = 8 \) and \( \Gamma_v = 5 \). We present a sensitivity study of these parameters in Section 6.3.

All computations were performed in Matlab 2012a on a 2.8 GHz Intel Core 2 Duo computer system with 8 GB memory.
Table 3 Optimal paths - WCVaR model using $q_{ij} = p_{ij}, d_{ij} = \pi(1.5)^2 \cdot \rho_{ij} - c_{ij}$, $\Gamma_u = 8$ and $\Gamma_v = 5$

<table>
<thead>
<tr>
<th>Confidence Level $\alpha$</th>
<th>Optimal WCVaR Route</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0.999933]</td>
<td>1,3,14,18,21,27,34,39,40,41,42,47,48,62,75,76,89,77,78,82,84</td>
</tr>
<tr>
<td>[0.999933, 0.999944]</td>
<td>1,3,14,18,21,27,34,39,40,41,42,47,48,62,75,76,89,77,78,82,84</td>
</tr>
<tr>
<td>[0.999944, 0.999955]</td>
<td>1,3,14,18,21,27,34,39,40,41,42,47,48,62,75,76,89,77,78,82,84</td>
</tr>
<tr>
<td>[0.999955, 0.999965]</td>
<td>1,3,14,18,21,27,34,39,40,41,42,47,48,62,75,76,89,77,78,82,84</td>
</tr>
<tr>
<td>[0.999965, 0.999968]</td>
<td>1,3,14,18,21,27,34,39,40,41,42,47,48,62,75,76,89,77,78,82,84</td>
</tr>
<tr>
<td>[0.999968, 0.999972]</td>
<td>1,3,14,18,21,27,34,39,40,41,42,47,72,73,74,48,62,75,76,89,77,78,82,84</td>
</tr>
<tr>
<td>[0.999972, 0.999973]</td>
<td>1,3,14,18,23,24,25,21,27,34,39,40,41,42,71,72,73,74,48,62,75,76,89,77,78,82,84</td>
</tr>
<tr>
<td>[0.999973, 0.999977]</td>
<td>1,3,14,18,23,24,25,21,27,34,39,40,41,42,71,72,73,74,48,62,75,76,89,77,78,82,84</td>
</tr>
<tr>
<td>[0.999977, 0.999986]</td>
<td>1,3,14,18,23,24,25,21,27,34,39,40,41,42,71,72,73,74,48,62,75,76,89,77,78,82,84</td>
</tr>
<tr>
<td>[0.999986, 0.999990]</td>
<td>1,3,14,18,23,24,25,21,27,34,39,40,41,42,71,72,73,74,48,62,75,76,89,77,78,82,84</td>
</tr>
<tr>
<td>[0.999990, 0.999993]</td>
<td>1,3,14,18,23,24,25,21,27,34,39,40,41,42,71,72,73,74,48,62,75,76,89,77,78,82,84</td>
</tr>
<tr>
<td>[0.999993, 0.999997]</td>
<td>1,4,3,14,17,28,35,27,34,39,43,38,85,54,67,69,80,70,83,84</td>
</tr>
<tr>
<td>(0.999997, 1)</td>
<td>1,3,14,18,21,27,34,39,43,38,85,54,67,69,80,70,83,84</td>
</tr>
</tbody>
</table>

Table 4 Optimal paths with $q_{ij} = p_{ij}, d_{ij} = \pi(1.5)^2 \cdot \rho_{ij} - c_{ij}$, $\Gamma_u = 8$, and $\Gamma_v = 5$

<table>
<thead>
<tr>
<th>Model</th>
<th>Optimal Route</th>
<th>Objective Function Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Traditional Risk (TR)</td>
<td>1,3,5,14,18,21,27,37,38,85,54,67,69,80,70,83,84</td>
<td>0.2076</td>
</tr>
<tr>
<td>Maximum Risk (MM)</td>
<td>1,3,5,14,18,21,27,34,39,43,38,85,54,67,69,80,70,83,84</td>
<td>0.7348</td>
</tr>
<tr>
<td>Worst-case TR (WTR)</td>
<td>1,3,4,14,18,21,27,34,39,40,41,42,47,48,62,75,76,89,77,78,82,84</td>
<td>17198</td>
</tr>
<tr>
<td>Worst-case MM (WMM)</td>
<td>1,3,5,14,18,21,27,34,39,43,38,85,54,67,69,80,70,83,84</td>
<td>38696</td>
</tr>
</tbody>
</table>

6.2. Comparison of Models

With the above data on hand, the WCVaR model resulted in 14 optimal paths for confidence levels in the interval [0, 1). The model was tested for 101 different confidence level values and the maximum computational time for a single $\alpha$-value was less than 21 seconds. We used CPLEX to solve the robust shortest-path sub-problems in the form of the MILP problem (32). The least-WCVaR paths are presented in Table 3 and sample paths graphically in Figures 4 and 5. We also presented the TR, MM, WTR, and WMM paths in Table 4.

An interesting observation from Table 3 is that the least-WCVaR path for $\alpha \geq 0.999997$ that is identical to the least-WMM path, as shown in Figure 4b, is passing through the highly populated area of downtown Buffalo. This is a rather unexpected result, since for confidence levels close to 1, the level of robustness is increasing; therefore, one would expect that the proposed path would avoid high populated areas for high $\alpha$-values. This behavior is due to the structure of the Buffalo network. Note that the path has its one side next to Niagara River. Hence the area affected by a potential accident along that path is cut to half, because we considered population exposure only in our current computation of accident consequence. This result encourages us to consider more flexible routing methods than simple methods like the MM or WMM models. Although the downtown area faces the waterway, it would still be unsafe to transport hazmat through the downtown area. Therefore, a decision maker would need alternatives, which the CVaR and WCVaR models can provide.
Figure 4 demonstrates the proposed paths from the WTR and WMM models for comparison purposes. As stated in Proposition 10, WCVaR for small values of $\alpha$ is equivalent with the WTR model. Figure 4a illustrates that the paths from the two models are indeed same. Similarly, the WCVaR path for large values of confidence level—specifically for $\alpha \geq 0.999997$—is equivalent to the WMM path (proposing the same path, as shown in Figure 4b), verifying Proposition 11. An interesting result is that the least-MM path and least-WMM path are identical. This result is specific to the Buffalo network, and cannot be generalized.

From the results, the flexibility of the WCVaR model is apparent. When the WTR or WMM models are used, the decision makers have a unique path in hand with no other alternatives to choose from, constraining their flexibility. On the contrary, using the WCVaR model, the decision maker has the ability, by altering the confidence level value, to obtain different paths covering various levels of risk-attitudes.

Let us also compare models with deterministic risk measures: TR, MM, VaR and CVaR. Figure 6 displays some routes used to validate the propositions regarding VaR and CVaR models. The TR path and the CVaR path with $\alpha = 0$ are identical as in Figure 6a, confirming Proposition 5. Note
that the VaR path with $\alpha = 0$, presented in Figure 6b, is different from the TR path, as expected. The VaR path in Figure 6b happened to be very similar to the TR path; this path is simply an output of the algorithm and all paths in the network have zero VaR value. These results confirm that CVaR offers risk-neutrality while VaR offers risk-indifference when $\alpha = 0$. If we compare the MM path shown in Figure 6c with the VaR and CVaR paths using large values for the confidence level, $\alpha$ close to 1, we can confirm Propositions 1 and 6.

Table 5 presents a more thorough comparison among various models with respect to various risk measure values and the length of paths. We also observe in Table 5 that, at $\alpha = 0.999995$, the VaR path produces relatively higher values in most risk measures, compared to other $\alpha$ values. This result indicates that the VaR model may produce an undesirable path by ignoring long tail. On the contrary, the behavior of the CVaR path at the same $\alpha$ value shows smooth changes. We can also confirm this point by comparing the VaR and CVaR models in terms of the CVaR risk measure. As shown in Figure 7a, the performance gap was as large as 17.6%.

We are also interested in how much value the WCVaR model would add over the VaR/CVaR models. If the CVaR model would already produce good paths in terms of the WCVaR risk measure, then it would weaken our motivation to consider the data uncertainty and tolerate computational complexity of the WCVaR model. Based on the data presented in Table 5, we provided a chart in Figure 7b. When the VaR and WCVaR paths are compared in terms of the WCVaR risk measure, the WCVaR paths showed significant improvements up to 34.5%. When compared with the CVaR paths, the WCVaR paths were better off by up to 17.4%. These observations signal that data uncertainty should not be overlooked.

Moreover, Table 5 provides us with information on the length of the paths proposed by each model. When WCVaR paths are compared with VaR and CVaR paths at the respective confidence
Table 5  Comparison of TR, MM, WTR, WMM, VaR, CVaR and WCVaR models, when \( q_{ij} = p_{ij}, 
\)
\[ d_{ij} = \pi (1.5) \cdot p_{ij} - c_{ij}, \] 
\( \Gamma_u = 8 \) and \( \Gamma_v = 5 \)

<table>
<thead>
<tr>
<th>Model</th>
<th>Confidence Level ( \alpha )</th>
<th>Risk Measure Values of each given path</th>
<th>Path Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>TR</td>
<td>MM</td>
<td>WTR</td>
</tr>
<tr>
<td>TR</td>
<td>0.2076</td>
<td>18033</td>
<td>0.7762</td>
</tr>
<tr>
<td>MM</td>
<td>0.2380</td>
<td>17198</td>
<td>0.8116</td>
</tr>
<tr>
<td>WTR</td>
<td>0.2399</td>
<td>18845</td>
<td>0.7348</td>
</tr>
<tr>
<td>WMM</td>
<td>0.2380</td>
<td>18033</td>
<td>0.8116</td>
</tr>
<tr>
<td>VaR</td>
<td>0</td>
<td>0.2126</td>
<td>18033</td>
</tr>
<tr>
<td></td>
<td>0.999970</td>
<td>0.2130</td>
<td>18033</td>
</tr>
<tr>
<td></td>
<td>0.999975</td>
<td>0.2076</td>
<td>18033</td>
</tr>
<tr>
<td></td>
<td>0.999980</td>
<td>0.2662</td>
<td>18845</td>
</tr>
<tr>
<td></td>
<td>0.999985</td>
<td>0.2076</td>
<td>18033</td>
</tr>
<tr>
<td></td>
<td>0.999990</td>
<td>0.2076</td>
<td>18033</td>
</tr>
<tr>
<td></td>
<td>0.999995</td>
<td>0.3279</td>
<td>27663</td>
</tr>
<tr>
<td></td>
<td>0.999999</td>
<td>0.2380</td>
<td>17198</td>
</tr>
<tr>
<td>CVaR</td>
<td>0</td>
<td>0.2076</td>
<td>18033</td>
</tr>
<tr>
<td></td>
<td>0.999970</td>
<td>0.2076</td>
<td>18033</td>
</tr>
<tr>
<td></td>
<td>0.999975</td>
<td>0.2081</td>
<td>18033</td>
</tr>
<tr>
<td></td>
<td>0.999980</td>
<td>0.2113</td>
<td>18033</td>
</tr>
<tr>
<td></td>
<td>0.999985</td>
<td>0.2739</td>
<td>18845</td>
</tr>
<tr>
<td></td>
<td>0.999990</td>
<td>0.2683</td>
<td>18845</td>
</tr>
<tr>
<td></td>
<td>0.999995</td>
<td>0.2536</td>
<td>18845</td>
</tr>
<tr>
<td></td>
<td>0.999999</td>
<td>0.2380</td>
<td>17198</td>
</tr>
<tr>
<td>WCVaR</td>
<td>0</td>
<td>0.2399</td>
<td>18845</td>
</tr>
<tr>
<td></td>
<td>0.999970</td>
<td>0.2378</td>
<td>18845</td>
</tr>
<tr>
<td></td>
<td>0.999975</td>
<td>0.2831</td>
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<td>0.999980</td>
<td>0.2683</td>
<td>18845</td>
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<tr>
<td></td>
<td>0.999985</td>
<td>0.3112</td>
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<td>0.3375</td>
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<td>0.2718</td>
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<tr>
<td></td>
<td>0.999970</td>
<td>0.2718</td>
<td>18845</td>
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<tr>
<td></td>
<td>0.999975</td>
<td>0.2831</td>
<td>18845</td>
</tr>
<tr>
<td></td>
<td>0.999980</td>
<td>0.2683</td>
<td>18845</td>
</tr>
<tr>
<td></td>
<td>0.999985</td>
<td>0.3112</td>
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</tr>
<tr>
<td></td>
<td>0.999990</td>
<td>0.3375</td>
<td>17198</td>
</tr>
<tr>
<td></td>
<td>0.999995</td>
<td>0.2466</td>
<td>17198</td>
</tr>
</tbody>
</table>

higher levels, the WCVaR paths are usually longer especially at lower confidence levels. We may interpret this observation as the robust nature of the WCVaR model induces more circuitous routes. When the CVaR path is already risk-averse with larger \( \alpha \) value, the consideration of data uncertainty impacts less.

6.3. Sensitivity Analysis

Risk-averse routing in hazmat transportation usually results in a very circuitous path, because one can usually avoid high accident consequences by detouring them. In this section, we discuss the relationship between the length of the least-WCVaR paths and various model parameters that are relevant to risk-averseness. In the WCVaR model, there are several parameters: \( \alpha, q_{ij}, p_{ij}, d_{ij}, c_{ij}, \Gamma_u \) and \( \Gamma_v \). Note that \( p_{ij} \) and \( c_{ij} \) are obtained from the understudy network, therefore, the decision maker has no control over them. On the other hand, we can control the level of robustness by altering the values of \( \alpha, q_{ij}, d_{ij}, \Gamma_u \) and \( \Gamma_v \). Keeping this fact in mind, sensitivity analyses were conducted keeping \( p_{ij} \) and \( c_{ij} \) fixed, and changing the values of \( \alpha, q_{ij}, d_{ij}, \Gamma_u \) and \( \Gamma_v \).
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(a) Improvements from the least-VaR to the least-CVaR path, measured in CVaR$_\alpha$

(b) Improvements from the least-VaR and least-CVaR paths to the least-WCVaR path, measured in WCVaR$_\alpha$

Figure 7 Improvement in CVaR and WCVaR risk measures

We tested the WCVaR model using 40 combinations of $\alpha$, $\Gamma_u$ and $\Gamma_v$ and for each combination $q_{ij}$ was in the interval $[0, 3p_{ij}]$ with incremental step size 0.25$p_{ij}$, and $d_{ij}$ was in the interval $[0, \pi(3)^2\rho_{ij} - c_{ij}]$ with incremental size 0.1 miles for the hazmat spread radius. For each combination of the parameter’s values we measured the length of the proposed path. In other words, we measure how circuitous the model becomes at various levels of robustness. A sample of the results is presented in Figure 8.

First, we observe that $q_{ij}$ and $d_{ij}$ shows no significant or unilateral relationship with the path length. At certain $\alpha$, $\Gamma_u$ and $\Gamma_v$ settings, there could be a positive trend (Figure 8a for $\alpha = 0.999970, \Gamma_u = 1, \Gamma_v = 1$), and at some other settings, there could be no trend (Figure 8b for $\alpha = 0.999975, \Gamma_u = 5, \Gamma_v = 8$). However, looking at the scale of the path-length-axis, we can observe that there are significant jumps from Figure 8a to Figure 8b. Motivated by this observation, we
generated Figure 8c to see if there exist a trend on the average path length for each pair of \( q \) and \( d \). As shown by Figure 8c no trend can be identified on the average path length.

Figure 9 demonstrates the relationship between the budgets of uncertainty \( \Gamma_u \) and \( \Gamma_v \), the confidence level \( \alpha \) and the travel distance of the proposed path. These plots were obtained by setting \( \Gamma_u \) equal to \( \Gamma_v \) in the interval \([0, 8]\) and varying \( \alpha \) values from 0.999950 to 0.999999. The vectors \( q_{ij} \) and \( d_{ij} \) were altered as described previously. As shown in Figures 9a and 9b there exist a positive trend between the uncertainty budgets and confidence level with the path length. That is up to the point where the confidence level is approaching 1. At that point due to the special structure of the understudy network the proposed path length is shorter than before. Figure 9c presents the relation between \( \Gamma_u \) and \( \Gamma_v \), the confidence level \( \alpha \) and the average travel distance of the proposed path for each combination. From the latter figure we can generalize our observation that the path becomes more circuitous with larger \( \Gamma_u \) (= \( \Gamma_v \)) values. This is also true with larger \( \alpha \); however, when \( \alpha \) is very large and approaching 1, the least-WCVaR path becomes short. This is because the understudy Buffalo network has waterfronts and only one arc becomes critical when \( \alpha \) is very large.

7. Conclusions

In this paper, we have formally introduced the notion of conditional value-at-risk (CVaR) and worst-case CVaR (WCVaR) to determine safe routes to transport hazardous materials (hazmat). We emphasize that CVaR and WCVaR are coherent risk measures and appropriate hazmat routing criteria. In addition, CVaR and WCVaR offer flexible, risk-averse, and computationally tractable routing methods. While CVaR and WCVaR violate the path selection optimality principle of Erkut and Verter (1998), we devised a tractable computational method to determine the least-CVaR/WCVaR paths. We confirmed our analytical findings via numerical experiments in a realistic road network in Buffalo, New York.
Our findings can be summarized as follows. The CVaR model offers flexible solutions varying from risk-neutral to extremely risk-averse. Although this observation is valid for general CVaR measures, it has more important meanings in the context of hazmat transportation. First, covering from the risk-neutral attitude is important. A confidence level 0.999 that looks safe enough would be inappropriate in real hazmat applications. The value-at-risk (VaR) model may provide a risk-indifferent solution under such a confidence level, which is unacceptable for hazmat routing. However, using the CVaR model, a decision maker can obtain the least-expected-risk path even with such a misuse of the confidence level. Therefore, covering from the risk-neutral risk attitude is important. Second, being risk-averse always and ignoring risk-seeking attitudes is completely fine in hazmat applications. Focusing only on the long tail to be risk-averse may be undesirable in certain financial investment problems, because high-risk could mean high-return. However, in the hazmat transportation, the return is always same: a safe shipment. This makes CVaR more attractive.

The WCVaR model inherits the analytical and computational characteristics of the CVaR model: WCVaR is a coherent and appropriate risk measure for hazmat routing, violating the path selection optimality principle. In addition, the WCVaR model considers the unique challenge of data uncertainty. The two important data in hazmat transportation are accident probability and accident consequence, both of which are usually insufficient. In addition, the nature of these two data sets is very different and the sources are usually independent. This property forces us to consider robust optimization problems with two uncertain multiplicative cost coefficients. We provided a computational method to solve the WCVaR minimization problem with such unique characteristics.

In general, the more risk-averse hazmat carrier is, the more circuitous path he/she chooses. This is natural that one could usually avoid highly populated areas by detouring city centers. Our experiences mostly agree with this consensus. However, in extreme cases like when the maximum risk model was used or CVaR/WCVaR models were used with a very large confidence level, the consensus may fail to be true. In the Buffalo network that we considered in this paper, those extreme risk models provided shorter routes that pass the central business area in the downtown. This was mainly because the Buffalo downtown area faces waterways of Niagara River, which reduces the accident consequences in the area by half. However, driving through the downtown area still would be an invalid idea for hazmat carriers. This observation in fact makes the flexibility of the CVaR/WCVaR models more attracatable so that a decision maker could easily generate alternatives.

A future direction to extend the results of this paper would be to consider different types of uncertainty sets. While we considered box-constrained uncertainty sets in this paper, one may also consider for example ellipsoidal uncertainty sets. It is known that the robust shortest-path problem itself becomes NP-hard with such sets; therefore one would need to develop an approximate or heuristic method to solve the corresponding WCVaR minimization problem. The challenge from the multiplicative two uncertain cost coefficients should also be addressed.
Endnotes
1. An earlier exposition of the CVaR concept in hazmat transportation appeared in a conference proceeding (Kwon 2011) with a couple of errors and a summary of the CVaR concept appeared in a handbook chapter (Toumazis et al. 2013) without mathematical rigorousness. This current paper introduces correct algorithms and analyses formally for both CVaR and WCVaR for the first time.

References


Proofs of Propositions and Lemmas

EC.1. Proof of Proposition 1

Proof: Suppose that $\alpha \to 0$. Then it is obvious, from the definition of VaR, that

$$\text{VaR}_0^l = \min \{ \beta : \Pr(R_l^l > \beta) \leq 1 \} = 0$$

for all path $l \in \mathcal{P}$.

Now suppose that $\alpha \to 1$. Then we have

$$\text{VaR}_1^l = \min \{ \beta : \Pr(R_l^l \leq \beta) \geq 1 \}$$

$$= \min \{ \beta : \Pr(R_l^l \leq \beta) = 1 \}$$

$$= C^l_{(m_l)}$$

where $C^l_{(m_l)}$ is the $m_l$-th smallest value in the set $\{ c_{ij} : (i, j) \in \mathcal{A}^l \}$ and $m_l = |\mathcal{A}^l|$ (the cardinality of $\mathcal{A}^l$). Therefore, we have

$$\text{VaR}_1^l = \max_{(i,j) \in \mathcal{A}^l} c_{ij}$$

which completes the proof. \(\square\)

EC.2. Proof of Proposition 2

Proof: We provide a proof for each axiom of CA1', CA2', and CA3'.

(CA1') Letting $r = s - m$, we obtain the desired result.

(CA2') Let

$$s^* = \arg \min_{s \in \mathbb{R}^+} \left[ s + \frac{1}{1-\alpha} \sum_{(i,j) \in \mathcal{A}^l_1} p_{ij} [c_{ij} - s]^+] \right]$$

and

$$t^* = \arg \min_{t \in \mathbb{R}^+} \left[ t + \frac{1}{1-\alpha} \sum_{(i,j) \in \mathcal{A}^l_2} p_{ij} [c_{ij} - t]^+] \right]$$

Then

$$\min_{r \in \mathbb{R}^+} \left[ r + \frac{1}{1-\alpha} \left\{ \sum_{(i,j) \in \mathcal{A}^l_1} p_{ij} [c_{ij} - r]^+ + \sum_{(i,j) \in \mathcal{A}^l_2} p_{ij} [c_{ij} - r]^+ \right\} \right]$$

$$\leq s^* + t^* + \frac{1}{1-\alpha} \left\{ \sum_{(i,j) \in \mathcal{A}^l_1} p_{ij} [c_{ij} - s^* - t^*]^+ + \sum_{(i,j) \in \mathcal{A}^l_2} p_{ij} [c_{ij} - s^* - t^*]^+ \right\}$$

$$\leq s^* + t^* + \frac{1}{1-\alpha} \left\{ \sum_{(i,j) \in \mathcal{A}^l_1} p_{ij} [c_{ij} - s^*]^+ + \sum_{(i,j) \in \mathcal{A}^l_2} p_{ij} [c_{ij} - t^*]^+ \right\}$$

$$= \min_{s \in \mathbb{R}^+} \left[ s + \frac{1}{1-\alpha} \sum_{(i,j) \in \mathcal{A}^l_1} p_{ij} [c_{ij} - s]^+] + \min_{t \in \mathbb{R}^+} \left[ t + \frac{1}{1-\alpha} \sum_{(i,j) \in \mathcal{A}^l_2} p_{ij} [c_{ij} - t]^+ \right]$$

(CA3') Letting $\lambda r = s$, we obtain the desired result. \(\square\)
EC.3. Proof of Proposition 3

Proof: We provide a proof for each axiom.

(HA1) We have

\[
\text{CVaR}_\alpha^{l2} = \min_{s \in \mathbb{R}^+} \left[ s + \frac{1}{1-\alpha} \sum_{(i,j) \in A^{l2}} p_{ij} [c_{ij} - s]^+ \right]
\]

\[
= \min_{s \in \mathbb{R}^+} \left[ s + \frac{1}{1-\alpha} \sum_{(i,j) \in A^{l1}} p_{ij} [c_{ij} - s]^+ + \frac{1}{1-\alpha} \sum_{(i,j) \in A^{l2} \setminus A^{l1}} p_{ij} [c_{ij} - s]^+ \right]
\]

\[
\geq \min_{t \in \mathbb{R}^+} \left[ t + \frac{1}{1-\alpha} \sum_{(i,j) \in A^{l1}} p_{ij} [c_{ij} - t]^+ + \frac{1}{1-\alpha} \min_{r \in \mathbb{R}^+} \left[ \sum_{(i,j) \in A^{l2} \setminus A^{l1}} p_{ij} [c_{ij} - r]^+ \right] \right]
\]

\[
= \min_{t \in \mathbb{R}^+} \left[ t + \frac{1}{1-\alpha} \sum_{(i,j) \in A^{l1}} p_{ij} [c_{ij} - t]^+ \right]
\]

\[
= \text{CVaR}_\alpha^{l1}
\]

where we set \( r \) an arbitrarily large number to make the corresponding minimum value zero.

(HA3) Let us introduce \( t^* \) such that

\[
\text{CVaR}_\alpha^{l1}(p, c) = \min_{t \in \mathbb{R}^+} \left[ t + \frac{1}{1-\alpha} \sum_{(i,j) \in A^{l1}} p_{ij} [c_{ij} - t]^+ \right]
\]

\[
= t^* + \frac{1}{1-\alpha} \sum_{(i,j) \in A^{l1}} p_{ij} [c_{ij} - t^*]^+
\]

and \( s^* \) such that

\[
\text{CVaR}_\alpha(p + m, c + n) = \min_{s \in \mathbb{R}^+} \left[ s + \frac{1}{1-\alpha} \sum_{(i,j) \in A^{l1}} (p_{ij} + m_{ij}) [c_{ij} + n_{ij} - s]^+ \right]
\]

\[
= s^* + \frac{1}{1-\alpha} \sum_{(i,j) \in A^{l1}} (p_{ij} + m_{ij}) [c_{ij} + n_{ij} - s^*]^+
\]

Then we obtain the following relations:

\[
\text{CVaR}_\alpha(p + m, c + n) = s^* + \frac{1}{1-\alpha} \sum_{(i,j) \in A^{l1}} (p_{ij} + m_{ij}) [c_{ij} + n_{ij} - s^*]^+
\]

\[
\geq s^* + \frac{1}{1-\alpha} \sum_{(i,j) \in A^{l1}} p_{ij} [c_{ij} - s^*]^+
\]

\[
\geq t^* + \frac{1}{1-\alpha} \sum_{(i,j) \in A^{l1}} p_{ij} [c_{ij} - t^*]^+
\]

\[
= \text{CVaR}_\alpha(p, c)
\]

\( \square \)
EC.4. Proof of Proposition 4

Proof: Let us consider any binary \( x' \), and the following problem:

\[
\min_{r \in \mathbb{R}^+} \left( r + \frac{1}{1 - \alpha} \sum_{(i,j) \in A} p_{ij} (c_{ij} - r)^+ x'_{ij} \right)
\]  

(EC.1)

Let \( \hat{C} \) be an ordered set of \( \{0\} \cup \{c_{ij} \forall (i, j) \in A\} \) and \( \hat{c}_k \) denote \( k \)-th smallest element of \( \hat{C} \), where \( \hat{c}_0 = 0 \). Then the problem (EC.1) is equivalent to the following problem:

\[
\min_{k \in \{0, 1, \ldots, |A|-1\}} \min_{r_k \in [\hat{c}_k, \hat{c}_{k+1}]} \left( r_k + \frac{1}{1 - \alpha} \sum_{(i,j) \in A \setminus \{r_k\}} p_{ij} (c_{ij} - r_k)^+ x'_{ij} \right)
\]  

(EC.2)

where \( A \setminus \{r_k\} = \{(i,j) : c_{ij} > r_k\} \). Note that we eliminated \( + \) operators. For each \( k \), the inner problem of \( r \) in (EC.2) is a linear optimization problem with a simple closed interval constraint. Therefore, \( r_k^* \) is either \( \hat{c}_k \) or \( \hat{c}_{k+1} \). Since \( x^* \) to the CVaR minimization problem (15) should be binary and the choice of \( x' \) is arbitrary, we obtain the proposition.

EC.5. Proof of Proposition 5

We first provide the following useful lemma:

**Lemma EC.1.** For any \( \alpha \in (0, 1) \) and path \( l \in \mathcal{P} \), if \( \text{VaR}^l_\alpha = 0 \), then \( \text{CVaR}^l_\alpha = \sum_{(i,j) \in A_l} p_{ij} c_{ij} \).

Proof: Consider a path \( l \in \mathcal{P} \) with \( \text{VaR}^l_\alpha = 0 \). Then we have,

\[
\lambda^l_\alpha = \Pr[R^l \leq \text{VaR}^l_\alpha] = \Pr[R^l \leq 0] = 1 - \sum_{(i,j) \in A_l} p_{ij}
\]

Therefore, we obtain

\[
\text{CVaR}^l_\alpha = \lambda^l_\alpha \text{VaR}^l_\alpha + (1 - \lambda^l_\alpha) \mathbb{E}[R^l : R^l > \text{VaR}^l_\alpha] = \left( \sum_{(i,j) \in A_l} p_{ij} \right) \mathbb{E}[R^l : R^l > 0]
\]

Using conditional probabilities, we can write:

\[
\mathbb{E}[R^l] = \mathbb{E}[R^l : R^l > \text{VaR}^l_\alpha] P(R^l > \text{VaR}^l_\alpha) + \mathbb{E}[R^l : R^l = \text{VaR}^l_\alpha] P(R^l = \text{VaR}^l_\alpha)
\]

\[
= \mathbb{E}[R^l : R^l > 0] P(R^l > 0) + \mathbb{E}[R^l : R^l = 0] P(R^l = 0)
\]

Since \( \mathbb{E}[R^l : R^l = 0] = 0 \), we obtain

\[
\mathbb{E}[R^l : R^l > 0] = \frac{\mathbb{E}[R^l]}{P(R^l > 0)} = \frac{\sum_{(i,j) \in A_l} p_{ij} c_{ij}}{\sum_{(i,j) \in A_l} p_{ij}}
\]

Hence, \( \text{CVaR}^l_\alpha \) is

\[
\text{CVaR}^l_\alpha = \left( \sum_{(i,j) \in A_l} p_{ij} \right) \frac{\sum_{(i,j) \in A_l} p_{ij} c_{ij}}{\sum_{(i,j) \in A_l} p_{ij}} = \sum_{(i,j) \in A_l} p_{ij} c_{ij}
\]

which completes the proof.

The main proof of Proposition 5 follows:
Proof: Let us define
\[ \alpha^l = 1 - \sum_{(i,j) \in A^l} p_{ij} \quad \text{and} \quad \alpha_{\min} = \min_{l \in P} \alpha^l \]
Then, \( \text{VaR}^l_{\alpha} = 0 \) for all \( \alpha \in (0, \alpha^l] \) (Kang et al. 2011, Theorem 1) and for each \( l \in \mathcal{P} \). Therefore, \( \text{VaR}^l_{\alpha} = 0 \) for all \( \alpha \in (0, \alpha_{\min}] \) and for all path \( l \in \mathcal{P} \). Then, by Lemma EC.1,
\[ \text{CVaR}^l_{\alpha} = \sum_{(i,j) \in A^l} p_{ij} c_{ij} \]
for all \( \alpha \in (0, \alpha_{\min}] \) and for all path \( l \in \mathcal{P} \). Therefore minimizing the CVaR measure is identical to minimizing the Traditional Risk measure. \( \square \)

EC.6. Proof of Proposition 6

We first provide the following useful lemma:

**Lemma EC.2.** For any \( \alpha \in (0, 1) \) and path \( l \in \mathcal{P} \), if \( \text{VaR}^l_{\alpha} = \max_{(i,j) \in A^l} c_{ij} \), then \( \text{CVaR}^l_{\alpha} = \text{VaR}^l_{\alpha} = \max_{(i,j) \in A^l} c_{ij} \).

**Proof:** Consider a path \( l \in \mathcal{P} \) with \( \text{VaR}^l_{\alpha} = \max_{(i,j) \in A^l} c_{ij} \). Then we have,
\[ \lambda^l_{\alpha} = \Pr \left[ R^l \leq \text{VaR}^l_{\alpha} \right] = \Pr \left[ R^l \leq \max_{(i,j) \in A^l} c_{ij} \right] = 1 \]
Therefore, we obtain
\[ \text{CVaR}^l_{\alpha} = \lambda^l_{\alpha} \text{VaR}^l_{\alpha} + (1 - \lambda^l_{\alpha}) E \left[ R^l : R^l > \text{VaR}^l_{\alpha} \right] = \text{VaR}^l_{\alpha} = \max_{(i,j) \in A^l} c_{ij} \]
Hence proof. \( \square \)

The main proof of Proposition 6 follows:

**Proof:** Kang et al. (2011, Theorem 2) showed that for each path \( l \in \mathcal{P} \), \( \text{VaR}^l_{\alpha} = \max_{(i,j) \in A^l} c_{ij} \) for all \( \alpha \in (p_{\text{max}}^l, 1) \). Therefore, by Lemma EC.2, we have \( \text{CVaR}^l_{\alpha} = \max_{(i,j) \in A^l} c_{ij} \) for all \( \alpha \in (p_{\text{max}}^l, 1) \) and for each path \( l \in \mathcal{P} \).

Consequently,
\[ \text{CVaR}^l_{\alpha} = \max_{(i,j) \in A^l} c_{ij} \]
for all \( \alpha \in (\alpha_{\text{max}}, 1) \) and for all path \( l \in \mathcal{P} \) where \( \alpha_{\text{max}} = \max_{l \in \mathcal{P}} p_{\text{max}}^l \). Therefore, the CVaR minimization is equivalent to the Maximum Risk minimization. \( \square \)
EC.7. Proof of Proposition 8

Proof: The coherence of WCVaR is assured by Proposition 7. We prove the rest.

(HA1) When path $l_2$ contains path $l_1$, we have

$$W_{CVaR}^0(U_p, V_c) = \min_{r \in \mathbb{R}^+} \max_{\hat{p} \in U_p, \tilde{c} \in V_c} \left\{ r + \frac{1}{1-\alpha} \sum_{(i,j) \in \mathcal{A}'^2} \hat{p}_{ij} [\hat{c}_{ij} - r]^+ \right\}$$

$$= \min_{s \in \mathbb{R}^+} \max_{\hat{p} \in U_p, \tilde{c} \in V_c} \left\{ s + \frac{1}{1-\alpha} \sum_{(i,j) \in \mathcal{A}'^1} \hat{p}_{ij} [\hat{c}_{ij} - s]^+ \right\}$$

$$\leq \min_{s \in \mathbb{R}^+} \max_{\hat{p} \in U_p, \tilde{c} \in V_c} \left\{ s + \frac{1}{1-\alpha} \sum_{(i,j) \in \mathcal{A}'^1} \hat{p}_{ij} [\hat{c}_{ij} - t]^+ \right\}$$

$$= \min_{s \in \mathbb{R}^+} \max_{\hat{p} \in U_p, \tilde{c} \in V_c} \left\{ s + \frac{1}{1-\alpha} \sum_{(i,j) \in \mathcal{A}'^1} \hat{p}_{ij} [\hat{c}_{ij} - s]^+ \right\}$$

$$= W_{CVaR}^0(U_p, V_c)$$

where we set $t$ an arbitrarily large number.

(HA3) For both cases of (HA3') and (HA3''), we can easily provide proofs that are similar to the CVaR measure case in Proposition 3.

\[\square\]

EC.8. Proof of Lemma 1

Proof: It is well known that a convex function retains its maximum at an extreme point if the feasible set is convex and compact (Pardalos and Rosen 1986). The objective function in (27) is a convex function of $v$ and the set $\mathcal{V}$ is a polytope. Therefore there exists a solution $v$ at an extreme point of $\mathcal{V}$, which is binary from the assumption that $\Gamma_v$ is a positive integer, for any given $r, x,$ and $u$.

\[\square\]

EC.9. Proof of Proposition 9

Proof: Let us consider any binary $x', u'$, and $v'$, and the following problem:

$$\min_{r \in \mathbb{R}^+} \left( r + \frac{1}{1-\alpha} \sum_{(i,j) \in \mathcal{A}} (p_{ij} + q_{ij} u'_{ij}) [(c_{ij} + d_{ij} v'_{ij}) - r]^+ x'_{ij} \right)$$  \hspace{1cm} (EC.3)

Let $\mathcal{R}$ be an ordered set of $\{0\} \cup \{c_{ij} : (i,j) \in \mathcal{A}\} \cup \{c_{ij} + d_{ij} : (i,j) \in \mathcal{A}\}$ and $r_k$ denote $k$-th smallest element of $\mathcal{R}$, where $\hat{r}_0 = 0$. Then the problem (EC.3) is equivalent to the following problem:

$$\min_{k \in \{0, 1, \ldots, |\mathcal{A}|-1\}} \min_{r_k \in [r_k, \hat{r}_{k+1}]} \left( r_k + \frac{1}{1-\alpha} \sum_{(i,j) \in \mathcal{A}(r_k)} (p_{ij} + q_{ij} u'_{ij}) [(c_{ij} + d_{ij} v'_{ij}) - r_k] x'_{ij} \right)$$  \hspace{1cm} (EC.4)
where \( A_1(r_k) = \{(i,j) : c_{ij} + d_{ij}v'_{ij} > r_k\} \). Note that we eliminated the ‘+’ operator in (EC.4). For each \( k \), the inner problem of \( r \) in (EC.4) is a linear optimization problem with a simple closed interval constraint. Therefore, \( r^*_k \) is either \( r_k \) or \( \hat{r}_{k+1} \). Since \( x^* \), \( u^* \), and \( v^* \) to the WCVaR minimization problem (26) are all binary and the choice of \( x', u', \) and \( v' \) are arbitrary, we obtain the theorem. \( \square \)

**EC.10. Proof of Proposition 10**

**Proof:** For any path \( l \), when \( \alpha \to 0 \), we have:

\[
WCVaR^l_{\alpha} = \min_{r \in \mathbb{R}^+} \left( r + \frac{1}{1-\alpha} \max_{\tilde{p} \in \mathcal{U}_p, \tilde{c} \in \mathcal{V}_c} \sum_{(i,j) \in A} \tilde{p}_{ij} [\tilde{c}_{ij} - r]^+ \right)
\]

\[
= \min_{r \in \mathbb{R}^+} \max_{\tilde{p} \in \mathcal{U}_p, \tilde{c} \in \mathcal{V}_c} \sum_{(i,j) \in A^l} \left( \frac{1}{|A^l|} r + \tilde{p}_{ij} [\tilde{c}_{ij} - r]^+ \right)
\]

Let us consider the two cases: when \( r > 0 \) and \( r = 0 \).

- **Case 1:** \( r > 0 \). In this case we have:

\[
\max_{\tilde{p} \in \mathcal{U}_p, \tilde{c} \in \mathcal{V}_c} \sum_{(i,j) \in A} \left( \frac{1}{|A^l|} r + \tilde{p}_{ij} [\tilde{c}_{ij} - r]^+ \right)
\]

\[
\geq \max_{\tilde{p} \in \mathcal{U}_p, \tilde{c} \in \mathcal{V}_c} \sum_{(i,j) \in A^l} \left( \frac{1}{|A^l|} r + \tilde{p}_{ij} [\tilde{c}_{ij} - r]^+ \right)
\]

\[
= \max_{\tilde{p} \in \mathcal{U}_p, \tilde{c} \in \mathcal{V}_c} \left( \sum_{(i,j) \in A^l} \tilde{p}_{ij} \tilde{c}_{ij} + \frac{1}{|A^l|} r \left( 1 - \sum_{(i,j) \in A^l} \tilde{p}_{ij} \right) \right)
\]

because \( 1 - \sum_{(i,j) \in A^l} \tilde{p}_{ij} > 0 \).

- **Case 2:** \( r = 0 \). In this case we have:

\[
\max_{\tilde{p} \in \mathcal{U}_p, \tilde{c} \in \mathcal{V}_c} \sum_{(i,j) \in A^l} \left( \frac{1}{|A^l|} r + \tilde{p}_{ij} [\tilde{c}_{ij} - r]^+ \right)
\]

\[
= \max_{\tilde{p} \in \mathcal{U}_p, \tilde{c} \in \mathcal{V}_c} \sum_{(i,j) \in A^l} \tilde{p}_{ij} \tilde{c}_{ij}
\]

Comparing the two cases, we can conclude that the optimal solution when \( \alpha \to 0 \) is given when \( r = 0 \), and the WCVaR model is equivalent to the worst-case TR model. \( \square \)

**EC.11. Proof of Proposition 11**

**Proof:** We consider

\[
WCVaR^l_{\alpha} = \min_{r \in \mathbb{R}^+} \left( r + \frac{1}{1-\alpha} \max_{\tilde{p} \in \mathcal{U}_p, \tilde{c} \in \mathcal{V}_c} \sum_{(i,j) \in A} \tilde{p}_{ij} [\tilde{c}_{ij} - r]^+ \right)
\]
which can be seen as a weighted sum of \( r \) and \( \max_{\tilde{p} \in U, \tilde{c} \in V} \sum_{(i,j) \in A} \tilde{p}_{ij} \tilde{c}_{ij} - r^+ \). When \( \alpha \to 1 \), the weight for the second term becomes very large and dominant. Therefore, to minimize the weighted sum, we can determine the solution

\[
 r^* = \max_{\tilde{c} \in V} \max_{(i,j) \in A} \tilde{c}_{ij}
\]

so that the second term vanishes. For any \( r \) greater than (EC.5), the second term vanishes, but the choice of (EC.5) also minimizes the first term. Hence,

\[
 WCVaR^*_l = \max_{\tilde{c} \in V} \max_{(i,j) \in A} \tilde{c}_{ij} = \max_{(i,j) \in A} (c_{ij} + d_{ij})
\]

for any \( l \in \mathcal{P} \) when \( \alpha \to 1 \).


Using the dual problem (32), Kwon et al. (2013) showed that a solution to the robust shortest path problem (30) could be obtained by solving finite number of (nominal) shortest path problems. To apply the result in the WCVaR minimization problem, let us define the following problem with an arbitrary set \( \Theta \):

\[
 Z(\Theta, r) = \min_{x \in \Omega, (\theta_u, \theta_v) \in \Theta} \Gamma_u \theta_u + \Gamma_v \theta_v + \sum_{(i,j) \in A} (p_{ij} e_{ij}(r) x_{ij} + \rho_{ij} + \mu_{ij})
\]

and the following sets:

\[
 \Theta_1(r) = \left\{ (\theta_u, \theta_v) : \theta_u \in \{0\} \cup \{q_{ij} e_{ij}(r) + q_{ij} f_{ij}(r), q_{ij} e_{ij}(r) : (i,j) \in A\},\right. \\
\left. \theta_v \in \{0\} \cup \{p_{ij} f_{ij}(r), p_{ij} f_{ij}(r) + q_{ij} f_{ij}(r) : (i,j) \in A\} \right\}
\]

\[
 \Theta_2(r) = \left\{ (\theta_u, \theta_v) : \theta_u \in \{0\} \cup \{q_{ij} e_{ij}(r) + q_{ij} f_{ij}(r), q_{ij} e_{ij}(r) : (i,j) \in A\},\right. \\
\left. \theta_u + \theta_v \in \{0\} \cup \{p_{ij} f_{ij}(r) + q_{ij} e_{ij}(r) + q_{ij} f_{ij}(r) : (i,j) \in A\}, \theta_v \geq 0 \right\}
\]

\[
 \Theta_3(r) = \left\{ (\theta_u, \theta_v) : \theta_v \in \{0\} \cup \{p_{ij} f_{ij}(r), p_{ij} f_{ij}(r) + q_{ij} f_{ij}(r) : (i,j) \in A\},\right. \\
\left. \theta_u + \theta_v \in \{0\} \cup \{p_{ij} f_{ij}(r) + q_{ij} e_{ij}(r) + q_{ij} f_{ij}(r) : (i,j) \in A\}, \theta_u \geq 0 \right\}
\]

Kwon et al. (2013, Theorem 2) shows that the robust shortest path problem (30) is equivalent to the following problem:

\[
 Z^*(r) = \min \{ Z(\Theta_1(r), r), Z(\Theta_2(r), r), Z(\Theta_3(r), r) \}
\]

(EC.7)
for any given \( r \). For methods to reduce the computational effort, see Kwon et al. (2013).

In support of Proposition 9 and (EC.7), the WCVaR minimization problem (26) is equivalent to the following problem:

\[
\text{WCVaR}_\alpha^* = \min_{r \in \mathcal{R}} \min \left\{ r + \frac{1}{1-\alpha} \mathcal{Z}(\Theta_1(r), r), \ r + \frac{1}{1-\alpha} \mathcal{Z}(\Theta_2(r), r), \ r + \frac{1}{1-\alpha} \mathcal{Z}(\Theta_3(r), r) \right\}
\]

where \( \mathcal{R} = \{0\} \cup \{c_{ij} : (i, j) \in \mathcal{A}\} \cup \{c_{ij} + d_{ij} : (i, j) \in \mathcal{A}\} \).