# Demand Learning and Dynamic Pricing under Competition in a State-Space Framework

**Chung, B.  Li, J.  Yao, T.  Kwon, C.  Friesz, T. L.**

## Abstract

In this paper, we propose a revenue optimization framework integrating demand learning and dynamic pricing for firms in monopoly or oligopoly markets. We introduce a state-space model for this revenue management problem, which incorporates game-theoretic demand dynamics and nonparametric techniques for estimating the evolution of underlying state variables. Under this framework, stringent model assumptions are removed. We develop a new demand learning algorithm using Markov chain Monte Carlo methods to estimate model parameters, unobserved state variables, and functional coefficients in the nonparametric part. Based on these estimates, future price sensitivities can be predicted, and the optimal pricing policy for the next planning period is obtained. To test the performance of demand learning strategies, we solve a monopoly firm's revenue maximizing problem in simulation studies. We then extend this paradigm to dynamic competition, where the problem is formulated as a differential variational inequality. Numerical examples show that our demand learning algorithm is efficient and robust.

## 1. INTRODUCTION

There has been an enormous amount of literature on dynamic pricing policy for revenue management (see [5], [15], [22]). The subject's popularity is largely because controlling price is an effective and direct way to manipulate market demand for services or products, so that a firm can maximize its profits in the short run. Thanks to rapidly growing information technology, we can conveniently gather, analyze and forecast market response or customers' behavior, and then update prices and inventories accordingly. As pointed out by Jayaraman and Baker [15], with the advent of the Internet and other more complex transaction formats, dynamic pricing is becoming more and more feasible and crucial in supporting the growth of many businesses. Therefore, how to learn and predict the impact of dynamic pricing decisions on the market and the competitors is the key to success in a variety of industries including the service industry (airlines, hotels, and rental car companies), the retail industry (department stores) and the e-commerce (see [4]). This requires us to carefully exploit efficient mathematical models and computational techniques in light of recent developments in statistics, optimization, and game theory.

In revenue management literature, many methods have been proposed to resolve the uncertainty of demands. They incorporated learning mechanisms either by experimentation or taking advantage of historical market data. Balvers and Cosimano [3] modeled demands as a linear function of prices with unknown slopes and intercepts, which motivated to learn by estimating parameters in the linear model. Mirman, Samuelson and Urbano [19] further examined the incentives of demand learning, and established two necessary conditions for a firm to learn uncertain demand curve from experiments. Later, Petruzzi and Dada [20] considered a demand model with both additive and multiplicative stochastic components, whose distributions are updated over time using Bayes' rule. Huang and Fang [14] incorporated a planned warranty term of products as a new factor in a demand function, and estimated uncertain parameters by market survey and analysis before utilizing a Bayesian decision model to determine the optimal warranty proportion in postsales service.

On the other hand, some researchers formulated demand dynamics from the perspective of customers' behavior. Gallego and van Ryzin [12] assumed that the number of customer arrivals has a Poisson distribution with exponentially distributed reservation prices in their mind. Under this assumption, optimal pricing strategy can be derived analytically. Aviv and Pazgal [2] and Araman and Caldentey [1] extended this idea to gamma distribution and two-point distribution, respectively. However, under this setting, beliefs about the distributions of several random variables have to be put *a priori*, and the impact of firms' historical prices was ignored.

Recent works on demand learning have begun to address the issue of competition. Bertsimas and Perakis [4] assumed that demand is a linear function of a firm's price and its competitors' prices, and estimated parameters using a least square method in cases of both monopoly and duopoly. Kwon et al. [18] considered dynamic games for demand learning, where the relationship between demand and price was characterized by evolutionary dynamics from the perspective of game theory. In their work, underlying price sensitivities were assumed to follow a random walk. Although this assumption guarantees a closed-form solution provided by Kalman filter, it is too restrictive and the whole algorithm will break down if this assumption is violated. Moreover, in practice, the model fails to capture future price sensitivity based on its patterns from the past.

In this paper, we propose a general framework for demand learning based on state-space models. The state-space model has been a powerful tool in modeling and forecasting dynamic systems, which was introduced by Kalman [16] and Kalman and Bucy [17]. It consists of an observation equation, which characterizes the dynamic of observed inputs and outputs, and a state equation, which describes the evolution of underlying unobserved state variables of the system we are interested in. For a state-space model with the linear state dynamics, Kalman filter yields good estimation and prediction. If the underlying state dynamics is not linear, the solution requires approximation or computation-intensive methods based on numerical integration. Pole and West [21] used Gaussian quadrature techniques in a Bayesian analysis of nonlinear dynamics models, and Carlin et al. [7] developed a Markov chain Monte Carlo (MCMC) approach for nonlinear and non-Gaussian state-space models.

When the price sensitivities in the demand function is considered as an unobservable state variable in a state-space model, the successes of demand learning and revenue maximization largely rely on estimating the pattern of the price sensitivities with high accuracy. The random walk assumption in Kwon et al. [18] implies that the historical price sensitivities provide no information about its future changes, since it is assumed that the price sensitivity at time $t$ equals to the price sensitivity at time $t-1$ plus Gaussian noise. Although this assumption provides analytical tractability, it may not be realistic in practice. In this paper, we greatly generalize this assumption by not making any assumption about the parametric form of the unobserved price sensitivity dynamics (i.e. the structure of the state equation) but learning it from the historical market data. Our method could discover the underlying patterns of the price sensitivities from the available market data, and automatically formulate the state equation that best describes how price sensitivities evolve over time. To be more precise, we incorporate a nonparametric functional-coefficient autoregressive (FAR) model to describe the nonlinear time series of the price sensitivities. This nonparametric technique relaxes parametric constraints, such that prior knowledge on the state equation structure is not required. Therefore, in our general state-space model, the observed demands and prices are described by a parametric observation equation, and the underlying state dynamics is captured by the nonparametric FAR model in the state equation. We develop a Bayesian method using MCMC algorithms to estimate model parameters, latent state variables, and functional-coefficients jointly. Then, we employ a simulated annealing algorithm for solving a single firm's pricing problem, and a fixed point algorithm for a non-cooperative competition problem.

The article is organized as follows. In section 2, we describe a revenue management model including demand dynamics, the evolution of underlying state variables, an optimal control formulation for a monopoly market, and a differential variational inequality (DVI) formulation for competition. In section 3, we explain the estimation and prediction procedures for the state-space model. Numerical examples and managerial implications for a monopoly and an oligopoly are presented in section 4. Finally, section 5 concludes the paper.

## 2. REVENUE MANAGEMENT MODEL

### 2.1 Demand Dynamics

We assume that customers are sensitive to the change of price. If there are multiple firms, customers are always searching for services or products at the most competitive prices. These so-called bargain-hunting buyers have no brand preference, and are willing to sacrifice some convenience for the sake of a lower price.

Following the game-theoretic dynamics proposed by Fudenberg and Levine [11], we assume that at time *t*, customers have a "reference price" in mind which reflects the market condition, and the demand at time *t* is a function of the difference between the current market price and the reference price. More precisely, the reference price $\tilde{\pi}_i$ is the weighted moving average price of past *k* time period of all firms:

$$\tilde{\pi}_i(t) = \int_{\tau=t-k}^{t} \sum_{f \in F} p_i^f(\tau) \pi_i^f(\tau), \quad (1)$$

where $F$ is the set of firms, $\tau \in [t-k, t]$ is the moving window, $\pi_i^f$ is the price of service *i* charged by firm *f*, and $p_i^f$ is the weight for $\pi_i^f$ with $\int_{\tau=t-k}^{t} \sum_{f \in F} p_i^f(\tau) = 1$. By choosing *k*, the impact significance of historical prices on the current market is specified. Then, from the perspective of the evolutionary game theory, the demand

$D_i^f(t)$ for the service type $i$ offered by a firm $f \in F$ evolves as follows

$$\frac{dD_i^f(t)}{dt} = \eta_i^f(t) \cdot (\tilde{\pi}_i(t) - \pi_i^f(t)). \quad (2)$$

The exogenous quantity $\eta_i^f$ could be interpreted as the price sensitivity of demand. It controls how quickly market demand reacts to price changes of service type $i$ from firm $f$. The firm estimates this unknown quantity by observing and analyzing the past market data.

This equation describes the relationship between observed demands and prices, and is usually called observation equation in a state-space framework. Since the demand dynamics is a function of firms' pricing strategy and consumers' price sensitivity, once price sensitivities over time are predicted, demand dynamics over a time interval will be determined by prices for the same period. Therefore, the revenue maximization problem reduces to an optimization problem over a closed set of prices.

### 2.2 Evolution of Price Sensitivity

Note that price sensitivity $\eta_i^f$ may exhibit periodic patterns like other time series in economics and business, or in general vary over time. For example, consumers may be less sensitive to price changes during Christmas holidays or other special events. Therefore, understanding its dynamics is a critical step in making pricing policy for future planning periods. Since price sensitivities cannot be directly observed, they are collectively called state variables in the state-space representation, and their evolution will be described by the state equation in our state-space model.

Many popular parametric time series structures can be used to describe the dependence of $\eta_i^f$ on its previous values, such as autoregressive moving-average

(ARMA) models, unit-root non-stationary random walk, Markov switching models and threshold autoregressive (TAR) models. However, in practice, we may not have sufficient knowledge to pre-specify a parametric form, and demand learning cannot be perfectly achieved by arbitrarily assuming a parametric structure. Moreover, the prediction performance is poor when the data is not actually driven by the model we specified.

Fortunately, recent developments of nonparametric techniques and computing facilities provide an alternative to model time series and relax parametric constraints, where no prior assumption of the model structure is required. Here we will use the functional-coefficient autoregressive (FAR) model proposed by Chen and Tsay [8], which proved robust against a range of underlying time series structures and is good at out-of-sample forecasting. Thus, the fluctuation of underlying state variables is captured by the state equation:

$$E(\eta_t) = f_1(\eta_{t-1},\cdots,\eta_{t-m_2})\eta_{t-1} + \cdots + f_{m_1}(\eta_{t-1},\cdots,\eta_{t-m_2})\eta_{t-m_1}, \quad (3)$$

where $(\eta_{t-1},\cdots,\eta_{t-m_2})$ is a vector of lagged values of $\eta_t$ and $f_j$, $j=1,\cdots,m_1$ are measurable functions from $\Re^{m_2}$ to $\Re^1$ assumed to be continuous and twice differentiable almost surely with respect to their arguments. The estimation of coefficient functions $f_1,\cdots,f_{m_1}$ from observed demands and prices allows appreciable flexibility on the structure of state equation. In fact, many popular linear or nonlinear parametric models are special cases of FAR model. Recently, the FAR model has been widely studied. To mention a few, Hoover et al. [13] developed the functional-coefficient model to longitudinal data. Cai et al. [6] applied the local linear regression method to estimate coefficient functions, which showed substantial improvements in post-sample forecasts over other parametric models. Tsay [23] further suggested fitting FAR models to discover nonlinear evolution of the state transition equation when specifying a nonlinear state-space model.

### 2.3 Optimal Control Problem for a Non-competitive Market

In this section, we provide an optimal control problem in a non-competitive market, which is used to compare the performance of various demand learning strategies in Section 5.1. The objective function of a firm is to maximize the net present value of revenue by providing a service with limited capacity over finite time horizon from $t_0$ to $t_f$. The firm's revenue at each time period can be calculated by multiplying the specified price and the realized demand. Nominal discount rate or interest rate (*r*) is used to compute the net present value of revenue.

The customer demand and the price sensitivity evolve over time according to the dynamics introduced in the previous sections. The price charged by this firm has upper and lower bounds due to market regulation, customer behaviors and firm's non-negligible cost. Moreover, the demand may be restricted by the non-negativity constraint and the limited capacity of the service provider. Consequently, a firm faces the following optimization problem:

$$\max_{\pi} \int_{t_0}^{t_f} e^{-rt}(\pi \cdot D)dt \qquad (4)$$

$$s.t.$$

$$\frac{dD}{dt} = \eta \cdot (\tilde{\pi} - \pi)$$

$$\pi_{\min} \le \pi \le \pi_{\max}$$

$$0 \le D \le D_{\max}.$$

where $\pi_{\max}$ and $\pi_{\min}$ are positive upper and lower bounds of price, respectively. $D_{\max}$ is the upper bound of demand.

## 2.4 Formulation for Competition

When competition among multiple firms offering multiple services is considered, the equilibrium problem can be formulated as a differential variational inequality or DVI [10]. The solution of DVI represents Cournot-Nash equilibria for the revenue-maximizing game of each firm. In this section, we present a DVI formulation for competition of multiple service providers. Since each firm $f$ in a market maximizes revenue, it has the following optimal control problem:

$$\max_{\pi^f} J_f(\pi^f, \pi^{-f}, t) = \int_{t_0}^{t_f} e^{-\rho t} (\sum_{i \in \mathrm{I}} \pi_i^f D_i^f) dt \quad (5)$$

$$s.t.$$

$$\frac{dD_i^f}{dt} = \eta_i^f \left( \frac{y_i}{|F|(t-t_0)} - \pi_i^f \right) \quad \forall i \in I \quad (6)$$

$$\frac{dy_i}{dt} = \pi_i^f + \sum_{g \in F/f} \pi_i^g \quad \forall i \in I \quad (7)$$

$$D_i^f(t_0) = K_{i,0}^f \quad \forall i \in I \quad (8)$$

$$y(t_0) = 0 \quad \forall i \in I \quad (9)$$

$$\pi_{\min,i}^f \le \pi_i^f \le \pi_{\max,i}^f \quad \forall i \in I \quad (10)$$

$$A_i^r D_i^f \le C_r^f \quad \forall r \in R \quad (11)$$

$$D_i^f \ge 0 \quad \forall f \in F, i \in I \quad (12)$$

In the objective function Eq. (5) of this model, prices are determined to maximize net profit value of revenue. Eq. (6)-(7) represent the demand dynamics with equally weighted average $p_\tau^f(t)$ for all $f$ at $t$. In other words, $p_\tau^f(t)$ is equal to $1/|F|(t-t_0)$ and the reference price becomes $\tilde{\pi}_i(t) = \int_{\tau=t-k}^{t} \sum_{f \in F} \pi_i^f(\tau)/|F|(t-t_0)$, where $|F|$ is the number of companies. By introducing a dummy variable $y_i(t)$, $\tilde{\pi}_i(t)$ can be written as $y_i(t)/|F|(t-t_0)$ together with Eq. (7). Initial values are considered in Eq. (8)-(9) and $K_{i,0}^f$ is the initial demand of service $i$ by firm $f$ **.** Eq. (10) ensures the price is bounded by its lower and upper limits. A joint resource constraint for firm $f$ is represented by Eq. (11), where $A$ is an incidence matrix showing the relationship between services and resources, and $C_r^f$ is the capacity of resource $r$ of firm $f$. The incidence matrix $A=(a_{ir})$ is defined as

$$a_{ir} = \begin{cases} 1 & \text{if resource r is used by service type i} \\ 0 & \text{otherwise.} \end{cases}$$

The last constraint Eq. (12) reflects non-negativity condition.

For the optimal control problem with fixed terminal time, Hamiltonian $H_f$ is defined as

$$\begin{aligned} &H_f(\pi^f; D^f, y; \lambda^f, \sigma^f; \alpha^f, \beta^f; \pi^{-f}, t) \\ &= e^{-\rho t}(\sum_{i\in \mathrm{I}} \pi_i^f D_i^f) + \Phi_f(\pi^f; D^f, y; \lambda^f, \sigma^f; \alpha^f, \beta^f; \pi^{-f}), \quad (13) \end{aligned}$$

where

$$\begin{aligned} &\Phi_f(\pi^f; D^f, y; \lambda^f, \sigma; \alpha, \beta; \pi^{-f}) \\ &= \sum_{i\in \mathrm{I}} \lambda_i^f \eta_i^f \left(\frac{y_i}{|F|(t-t_0)} - \pi_i^f\right) + \sum_{i\in \mathrm{I}} \sigma_i^f (\pi_i^f + \sum_{g\in F/f} \pi_i^g) + \sum_{i\in \mathrm{I}} \alpha_i^f(-D_i^f) + \sum_{s\in \mathrm{I}} \beta_s^f (A_i^s D_i^f - C_s^f). \quad (14) \end{aligned}$$

The two new variables, $\lambda_i^f$ and $\sigma_i^f$, are the adjoint vectors satisfying transversality conditions due to the free endpoint conditions, which are $\lambda_i^{f*}(t_f)=0$ and $\sigma_i^{f*}(t_f)=0$. Also, $\alpha_i^f$ and $\beta_i^f$ are dual variables for Eq. (11)-(12).

Now, we have the following DVI formulation and the solution represents Cournot-Nash equilibria for the revenue-maximizing game of each firm:

$$\int_{t_0}^{t_f} \left(\sum_{i\in I}\sum_{f\in F} \frac{\partial H_f^*}{\partial \pi_f}(\pi_i^f - \pi_i^{f*})\right) dt \le 0 \quad (15)$$

$$\text{for} \quad \forall \pi \in \Omega = \prod_{f\in F} \Omega_f$$

where

$\Omega_f = \{\pi^f : \pi^f_{\min,i} \le \pi^f_i \le \pi^f_{\max,i}\}$ and $H^*_f = H_f(\pi^{f*}; D^{f*}, y^*; \lambda^{f*}, \sigma^{f*}; \alpha^{f*}, \beta^{f*}; \pi^{-f*}, t)$.

## 3. ESTIMATION AND PREDICTION

So far, we have introduced a revenue management model for demand learning, where demand dynamics is described by a state-space model, and the optimal pricing policy can be obtained by solving the corresponding optimization problem. By making use of the historical data, we could estimate unknown quantities in the state-space model, and then forecast realized demands in the future following our optimization procedure.

However, since observations for model estimation occur only at discrete times, we first discretize a whole planning period into $K$ sub-intervals with the same length and suppose one observation is made at the end of each sub-interval. Then, the state-space model is reformulated as

$$\eta_t = \sum_{j=1}^{m_1} f_j(\eta_{t-1}, \cdots, \eta_{t-m_2})\eta_{t-j} + u_t,\ u_t \sim N(0, \sigma^2), \quad (16)$$

$$\Delta D_t = \eta_t \cdot (\tilde{\pi}_t - \pi_t) + v_t,\ v_t \sim N(0, \omega^2), \quad (17)$$

where $t \in \{1, 2, ..., K\}$, $f_j$, $j = 1, \cdots, m_1$ in the state equation are measurable functions, $u_t$ and $v_t$ are Gaussian noise with different variances. By specifying parametric forms for $f_j(\cdot)$'s, this model could be reduced to one of familiar time series models described in Section 2. On the other hand, we may wish to estimate these general forms of functional-coefficients by nonparametric techniques instead of imposing arbitrary constraints, such that underlying dynamics of price sensitivities can be precisely recovered.

At the end of one planning period, we estimate unknown quantities $\sigma^2, \omega^2, f_1, \cdots, f_{m1}$ and $\eta_1, \cdots, \eta_K$ using observed demands and prices in the past planning periods, forecast the dynamics of $\eta_t$ in the next planning period, and finally determine the firm's pricing policy for the next planning period to maximize the revenue.

### 3.1 MCMC Estimation for State-Space Model with AR(1) State Dynamics

Let us start from a parametric state equation by assuming an autoregressive state dynamics. That is, we assume that the dynamic of price sensitivity $\eta_k$ follows an AR(1) process. Then we have $m_1 = 1$, $m_2 = 1$ and $f_1 = \phi$. That is, the state equation is reduced to

$$\eta_t = \phi \eta_{t-1} + u_t,\ u_t \sim N(0, \sigma^2). \quad (18)$$

In our MCMC-implemented Bayesian estimation, we choose the following conjugate prior distributions for parameters in the state-space model: $\sigma^2 \sim IG(a_0, b_0)$ , $\omega^2 \sim IG(c_0, d_0)$, and $\phi \sim N(\mu_\phi, \sigma_\phi^2)$, where *IG* refers to inverse gamma distribution and *N* refers to a normal distribution. Then, at the *i*-th iteration, the MCMC steps are:

a) Initialize $\sigma^2, \omega^2, \phi, \eta_t$ for $t$ = 1, 2, … , K.

b) Update $\sigma^2$, variance of errors in the state equation. Since the data likelihood is normal

$$\eta_t \mid \eta_{t-1}, \sigma^2 \sim N\left(\phi \eta_{t-1}, \sigma^2\right),$$

and the prior distribution is conjugate inverse gamma, the conditional distribution of $\sigma^2$ is also inverse gamma. Therefore, conditioning on $\omega^{2(i-1)}$ , $\phi^{(i-1)}$ and

$\eta_1^{(i-1)},\cdots,\eta_K^{(i-1)}$ from the previous iteration, we draw a new sample $\sigma^{2(i)}$ from the following distribution

$$\sigma^2 \mid (.) \sim IG\left( a_0 + \frac{n}{2}, \left( \frac{1}{b_0} + \frac{1}{2} \sum (\eta_t^{(i-1)} - \phi^{(i-1)} \eta_{t-1}^{(i-1)})^2 \right)^{-1} \right).$$

c) Update $\omega^2$, variance of errors in the observation equation. Similarly, $\omega^{2(i)}$ is drawn from its posterior distribution

$$\omega^2 \mid (.) \sim IG\left( c_0 + \frac{n}{2}, \left( \frac{1}{d_0} + \frac{1}{2} \sum (\Delta D_t - \eta_t^{(i-1)} \cdot (\tilde{\pi}_t - \pi_t))^2 \right)^{-1} \right).$$

d) Update latent state variables $\eta_t$, for $t$ = 1, 2, … , K. Note that both nonlinear state equation and linear observation equation contain information about $\eta_t$. The likelihood of $\eta_t$ from the state equation is $\eta_{t+1} - \phi\eta_t \sim N(0,\sigma^2)$ and $\eta_t - \phi\eta_{t-1} \sim N(0,\sigma^2)$, and that from the observation equation is $\Delta D_t - \eta_t \cdot (\tilde{\pi}_t - \pi_t) \sim N(0,\omega^2)$, which on manipulation gives the following posterior distribution

$$\eta_t \mid (\cdot) \propto \omega(\eta_t) N(B_2 b_2, B_2),$$

where

$$B_2^{-1} = \frac{1}{\sigma^{2(i-1)}} + \frac{\phi^{2(i-1)}}{\sigma^{2(i-1)}}, \quad b = \frac{\phi\eta_{t-1}^{(i-1)}}{\sigma_\phi^2} + \frac{\phi\eta_{t+1}^{(i-1)}}{\sigma^{2(i-1)}},$$

$$\text{and} \quad \omega(\eta_t) = \exp\left\{ -\frac{1}{2\omega^{2(i-1)}} \left( \Delta D_t - \eta_t^{(i-1)} \cdot (\tilde{\pi}_t - \pi_t) \right)^2 \right\}.$$

Since this posterior distribution is not a closed form, we cannot directly draw a new sample $\eta_t^{(i)}$. So we use accept-reject algorithm or in general Metropolis-Hasting algorithm.

e) Update AR(1) coefficient $\phi$. The state equation gives $\eta_t \sim N(\phi\eta_{t-1},\sigma^2)$, therefore, by conjugate normal prior we specified in the initialization step, the posterior distribution of $\phi$ is also normal: $N(Bb, B)$, where $B^{-1} = \frac{1}{\sigma_\phi^2} + \sum_t \frac{\eta_{t-1}^{2(i-1)}}{\sigma^{2(i-1)}}$, and $b = \frac{\mu_\phi}{\sigma_\phi^2} + \sum_t \frac{\eta_t^{(i-1)}\eta_{t-1}^{(i-1)}}{\sigma^{2(i-1)}}$. We sample $\phi^{(i)}$ from this distribution.

f) Repeat b) – e).

Up to now, we have carried out one cycle of the MCMC and are ready to continue sampling for the next cycle. The sample process continues until the chains converge to the stationary distributions. We then collect all posterior samples and use their posterior medians as the point estimates of all parameters and latent state variables.

## 3.2 MCMC Estimation for State-Space Model with Functional-Coefficients

However, if we do not assume any parametric structure for $f_j(\cdot)$'s, nonparametric techniques such as kernel regression or local linear regression can be used to estimate the functional-coefficients $f_j(\cdot)$'s. In our example, we take $m_1 = 2$ and $m_2 = 1$ to avoid overfitting as suggested by Cai et al. [6]. In this way, price sensitivity at time *t*, $\eta_t$, is regressed on $\eta_{t-1}$ and $\eta_{t-2}$, and the functional regression coefficients only depend on $\eta_{t-1}$. The extension to cases $m_1 > 2$ and $m_2 > 1$ is straightforward. Therefore, the state equation in our state-space model becomes

$$\eta_t = f_1(\eta_{t-1})\eta_{t-1} + f_2(\eta_{t-1})\eta_{t-2} + u_t,\ u_t \sim N(0, \sigma^2). \quad (19)$$

In the presence of functional-coefficient, the MCMC estimating procedure is similar to the one we described above. However, the conditional posterior distribution of state variables $\eta_t$ becomes

$$L(\eta_t^{(i)} \mid .) \propto p(\eta_t^{(i)} \mid \eta_{t+1}^{(i-1)}, \eta_{t+2}^{(i-1)}, \sigma^{2(i-1)})\, p(\eta_t^{(i)} \mid \eta_{t-1}^{(i-1)}, \eta_{t+1}^{(i-1)}, \sigma^{2(i-1)})\, p(\eta_t^{(i)} \mid \eta_{t-1}^{(i-1)}, \eta_{t-2}^{(i-1)}, \sigma^{2(i-1)})$$
$$\cdot p(\eta_t^{(i)} \mid \Delta D_t, \Delta\pi_t, \omega^{2(i-1)}). \quad (20)$$

Since there is no corresponding closed form for this posterior distribution, we again employ Metropolis-Hasting algorithm to get posterior samples of $\eta_t^{(i)}$ that follow this distribution. According to the Metropolis-Hasting algorithm, at each cycle of the MCMC, we first draw a sample from a closed-form distribution, which is called proposal distribution, and then accept this sample with certain probability. After many iterations, the resulting posterior samples follow the desired distribution. In particular, at the *i*-th iteration, we draw a new sample of $\eta_t$ denoted by $\eta_t^*$ from a proposal distribution

$$q(\eta_t^* \mid \eta_{t-1}^{(i-1)}, \eta_{t-2}^{(i-1)}, \sigma^{2(i-1)}) \propto \exp\left\{-\frac{1}{2\sigma^{2(i-1)}}\left(\eta_t^* - f_1(\eta_{t-1}^{(i-1)})\eta_{t-1}^{(i-1)} - f_2(\eta_{t-1}^{(i-1)})\eta_{t-2}^{(i-1)}\right)^2\right\}, \quad (21)$$

and then accept $\eta_t^{(i)} = \eta_t^*$ with the probability $\min(1, p^*)$, where

$$p^* = \frac{L\left(\eta_t^* \mid .\right)}{L\left(\eta_t^{(i-1)} \mid .\right)} \times \frac{q\left(\eta_t^{(i-1)} \mid \eta_{t-1}^{(i-1)}, \eta_{t-2}^{(i-1)}, \sigma^{2(i-1)}\right)}{q\left(\eta_t^* \mid \eta_{t-1}^{(i-1)}, \eta_{t-2}^{(i-1)}, \sigma^{2(i-1)}\right)}. \quad (22)$$

If the new value $\eta_t^*$ is not accepted, we set $\eta_t^{(i)} = \eta_t^{(i-1)}$.

Moreover, we are not updating $\varphi$ here but updating more general two functions $f_1$ and $f_2$ by least square estimations based on $\eta_1^{(i)}, \cdots, \eta_t^{(i)}$ of current iteration. That is, we have

$$\hat{f}_j(u_0) = \sum_{i=1}^{K} K_{K,j}(X_i, \eta_{t-1} - u_0)\eta_t \quad j = 1, 2 \quad (23)$$

where

$$K_{K,j}(x,u) = e_{j,2}^T (\tilde{X}^T W \tilde{X})^{-1} \begin{pmatrix} x \\ ux \end{pmatrix} K_h(u). \quad (24)$$

In the above expression, $X_i = (\eta_{t-1}, \eta_{t-2})$, $K_h(\cdot)$ is a kernel function with bandwidth

$h$ selected by cross-validation, $e_{j,2}$ is the $4\times 1$ vector with 1 at the j-th and (2+j)-th positions, $\tilde{X}$ denotes an $K\times 4$ matrix with $\left(X_i^T, X_i^T(\eta_{i-1}-u_0)\right)$ as its i-th row, and $W$ is a $K\times K$ diagonal matrix with diagonal entries: $K_h(\eta_{i-1}-u_0)$.

Likewise, after the chains become stationary, we calculate the medians of all posterior samples as point estimates. By plugging in estimated parameters, estimated coefficient functions and estimated latent state variables of the past, future state variables can be predicted.

### 3.3 MCMC Estimation for Missing Data

In the above two algorithms, we assume that there is no missing data. That is, we observe $\Delta D_t$ and $\pi_t$ for $t$ = 1, 2, … , K. However, in practice, a subset of observations, say $\Delta D_{t'}$ and $\pi_{t''}$, may be missing. If statistical inference is easier when we have the complete data consisting of observed data and missing data, we may use a strategy called “data augment” to overcome the difficulties due to missing data. The essential idea here is to substitute the missing values with simulations, and then perform the standard estimation procedure using the augmented dataset. The strategy allows us to estimate all missing values, and facilitates the parameter estimations, since the proposed estimation procedure based on the complete data remains after data augment.

Assume that $\Delta D_{t'}$, $t'=v_1,\ldots,v_{n'}$, and $\pi_{t''}$, $t''=w_1,\ldots,w_{n''}$, are missing, that means, we have $n^{'}+n^{''}$ missing values. Before implementing MCMC algorithms, we may initialize missing observations. Then, at the $i$-th iteration of the MCMC sampling procedure described above, we draw a new sample $\Delta D_{t'}^{(i)}$ from its posterior distribution conditional on $\eta_t^{(i-1)}$ and $\pi_t^{(i-1)}$, $t$ = 1, 2, … , K,

$$\Delta D_{t'} \,|\, (.) \sim N(\eta_{t'}(\pi_{t'} - \pi_{t'}), \omega^2), \quad (25)$$

and draw $\pi_{t''}^{(i)}$ from its posterior distribution conditional on $\eta_t^{(i-1)}$ and $\Delta D_t^{(i-1)}$, $t$ = 1, 2, … , K,

$$\pi_{t''} \,|\, (.) \sim N(\pi_{t''} - \frac{\Delta D_{t''}}{\eta_{t''}}), \omega^2). \quad (26)$$

# 4. NUMERICAL EXAMPLES

## 4.1 A Single Firm's Problem

Let us first consider a monopoly market where a single firm provides a service. We assume that the true price sensitivity follows one of the following patterns: (a) random walk: $\eta_t = \eta_{t-1} + u_t$ (Kwon et al. [18]), (b) autoregressive structure of order 1: $\eta_t = 0.8\eta_{t-1} + u_t$, (c) autoregressive structure of order 2: $\eta_t = 0.8\eta_{t-1} + 0.4\eta_{t-2} + u_t$, (d) sine wave: $\eta_t = \sin(t) + 2 + u_t$ (Friesz et al. [10]), (e) composite sine wave: $\eta_t = \sin(t) + \sin(2t) + \sin(4t) + u_t$, (f) sawtooth wave: $\eta_t = \frac{1}{2} - \frac{2}{\pi}\sum_{j=1}^{50}\frac{1}{j}\sin(\frac{2\pi jt}{6}) + u_t$,

where $u_t$ ~ N(0, $\sigma^2$), and $\sigma^2$ = 0.2. Although these state dynamics of price sensitivities are unknown to the service provider, given historical demands and prices (see Figure 1), a firm can estimate their patterns and make forecasts using one of the following demand learning strategies: (a) Kalman filter for linear state dynamics proposed by Kwon et al. [18], (b) MCMC algorithm with assumed AR(1) dynamics and (c) MCMC algorithm with functional-coefficient autoregressive (FAR) model.

We assess the forecasting performance of different strategies by the average squared errors (ASE):

$$ASE = \frac{1}{n}\sum_{t=31}^{60}(\hat{\eta}_t - \eta_t)^2, \quad (27)$$

where $\hat{\eta}_t$ is the predicted price sensitivity in the next planning period (30 days). Table 1 gives ASE of each learning method. Since Kalman filter demand learning assumes a random walk dynamics of price sensitivity, it yields good forecasts only when the true price sensitivity indeed follows a random walk process. Similarly, MCMC demand learning with assumed AR(1) dynamics has good performance only when the underlying state dynamics follows a random walk or AR(1) process, since the AR(1) structure reduces to a random walk when the autoregressive parameter is close to 1. Therefore, it is clear that for parametric demand learning methods, to achieve good predictive performance, the assumed parametric structure of unobservable state dynamics should be correct. On the other hand, the nonparametric MCMC demand learning provides the most accurate forecasts for all underlying price sensitivity dynamics, since nonparametric technique could recover it nicely from the historical data without assuming state dynamics.

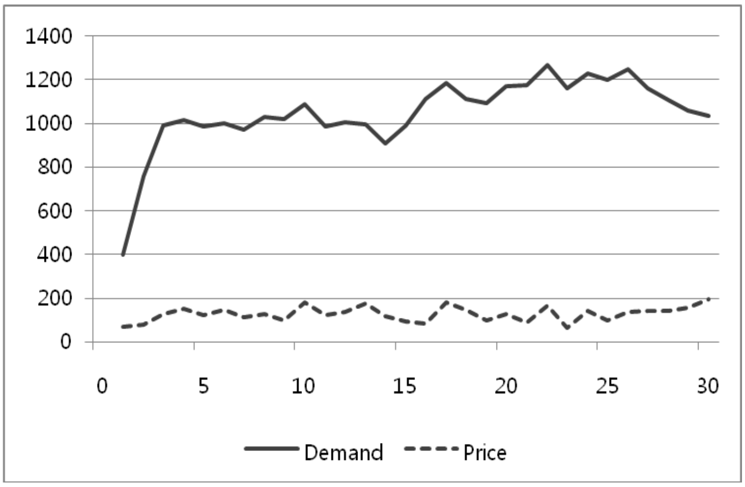


Figure 1. An example of history data

| Underlying Dynamics | Kalman Filter | MCMC with AR(1) | MCMC with FAR |
|---|---|---|---|
| Random walk | 0.0367 | 0.0352 | 0.0361 |
| AR(1) | 0.1277 | 0.0945 | 0.0831 |
| AR(2) | 0.1659 | 0.1209 | 0.0985 |
| Sine | 1.5451 | 0.6069 | 0.1466 |
| Composite sine | 2.8575 | 1.9332 | 0.9036 |
| Sawtooth wave | 0.4602 | 0.4421 | 0.2953 |

Table 1. Average squared errors of forecasts over 100 simulations

After predicting price sensitivity dynamics by one of the three strategies, optimal pricing policy for the future planning period is determined by simulated annealing algorithms. Then according to true underlying dynamics of $\eta_t$, demands corresponding to the pricing policies are observed respectively and realized revenues are calculated. We take the sine underlying dynamics as an example, and report the realized revenues in Table 2. In order to investigate the influence of noise, simulations are performed with $\sigma$ = 0, 0.1, 0.2, 0.3, 0.4, 0.5 and 1.

As we can see from Table 2, the Kalman filter method could not generate more revenue than the other learning methods, since its restrictive assumption of linear dynamics of state variables as discussed above, but nonparametric MCMC demand learning strategy significantly outperforms the others. As noise decreases, the average revenues of all demand learning methods increase, which implies that the overall accuracy of demand learning methods improves; as noise increases, the observed data includes a significant proportion of randomness, and thus it is very hard to recover the underlying dynamics from the data by any statistical demand learning method. Finally, when the noise is large enough (including $\sigma \geq 1$), revenues generated by three methods are similar since it is very difficult to extract pattern form the data. This simulation study in a non-competitive market demonstrates the motivation and importance of demand learning for dynamic pricing.

| $\sigma$ | Kalman Filter | MCMC with AR(1) | MCMC with FAR |
|---|---|---|---|
| 0 | 3,523,780 | 5,758,807 | 21,514,740 |
| 0.1 | 3,288,385 | 5,845,678 | 18,496,175 |
| 0.2 | 3,069,296 | 5,727,536 | 14,687,021 |
| 0.3 | 3,301,496 | 5,667,405 | 10,395,353 |
| 0.4 | 3,180,353 | 5,521,956 | 8,845,662 |
| 0.5 | 3,062,381 | 5,566,315 | 6,971,845 |
| 1 | 3,014,238 | 4,260,647 | 4,915,539 |

Table 2. Average realized revenue over 100 simulations

### 4.2 Multiple Firms' Problem - Competition

In cases of competition, one firm's demand and revenue are influenced by competing firms' pricing policy. Therefore, demand parameters for all firms in a market have to be estimated and forecasted simultaneously when demand learning based dynamic pricing is performed. In this section, it is assumed that the firms believe that competitors are also using the same learning strategy (e.g. Bertsimas and Perakis (2006)). Also, we assume that the market has reached equilibrium during the past planning period. With the historical market data, each firm can select one of the following pricing policies: (a) random pricing, (b) static pricing and (c) demand learning based dynamic pricing. A firm employing random pricing policy chooses time-varying random price within a feasible price set. For static pricing, the average value in a feasible price set is calculated and the single estimate is used during the next planning horizon. In the case of dynamic pricing, the MCMC algorithm with FAR model is considered as a demand learning strategy.

| Service Type ($i$) | | Service 1 | Service 2 | Service 3 | Service 4 |
|---|---|---|---|---|---|
| $K_{i,0}^{f}$ | Firm1 | 10.0 | 17.5 | 22.5 | 30.0 |
| | Firm2 | 9.5 | 16.5 | 20.0 | 31.0 |
| $\pi_{\max,i}^{f}$ | Firm1 | 85 | 135 | 180 | 205 |
| | Firm2 | 75 | 108 | 185 | 210 |
| $\pi_{\min,i}^{f}$ | Firm1 | 30 | 40 | 60 | 130 |
| | Firm2 | 45 | 50 | 65 | 115 |

Table 3. Service dependent data

| Resource Type ($r$) | | Resource 1 | Resource 2 | Resource 3 | Resource 4 | Resource 5 |
|---|---|---|---|---|---|---|
| $C_{r}^{f}$ | Firm1 | 300 | 210 | 150 | 60 | 255 |

| | | | | | | |
|---|---|---|---|---|---|---|
| | Firm2 | 180 | 150 | 120 | 75 | 210 |

Table 4. Resource dependent data

Our numerical example considers two firms with four services and five resources to illustrate the revenue maximization problem under competition. Table 3 summarizes the service dependent data including initial demands and price boundaries according to Friesz et al. [10]. Also, Table 4 shows the resource capacity for each firm. The incidence matrix between resources and services is given by

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

| Pricing policy | | $\sigma = 0$ (No noise) | | $\sigma = 0.1$ | |
|---|---|---|---|---|---|
| Firm1 | Firm2 | Firm1 | Firm2 | Firm1 | Firm2 |
| Random | Random | 1,798,357 | 1,241,279 | 1,768,015 | 1,142,184 |
| Dynamic | Random | 6,022,001 | 18,246 | 6,114,322 | 8,637 |
| Random | Dynamic | 122,380 | 5,338,070 | 89,291 | 5,283,634 |
| Static | Static | 1,513,300 | 851,200 | 1,317,772 | 787,619 |
| Dynamic | Static | 6,074,300 | 10,433 | 5,946,449 | 4,555 |
| Static | Dynamic | 108,350 | 5,336,600 | 75,112 | 5,412,261 |
| Dynamic | Dynamic | 811,130 | 259,560 | 748,021 | 239,033 |

| Pricing policy | | $\sigma = 0.2$ | | $\sigma = 0.3$ | |
|---|---|---|---|---|---|
| Firm1 | Firm2 | Firm1 | Firm2 | Firm1 | Firm2 |
| Random | Random | 1,694,559 | 1,228,460 | 1,643,022 | 1,299,857 |
| Dynamic | Random | 6,120,186 | 9,218 | 6,137,713 | 14,515 |
| Random | Dynamic | 100,092 | 5,449,133 | 101,644 | 5,570,901 |
| Static | Static | 1,436,475 | 763,137 | 1,442,174 | 776,509 |
| Dynamic | Static | 6,112,101 | 6,476 | 6,189,109 | 8,631 |
| Static | Dynamic | 82,501 | 5,536,100 | 83,322 | 5,556,947 |
| Dynamic | Dynamic | 867,265 | 382,695 | 914,466 | 421,076 |

Table 5. Average revenue of the firms

Table 5 shows the average revenues of 100 simulations for different combinations of pricing policies. We observe that when competitor's pricing strategy is fixed, demand learning based dynamic pricing is a better approach for a firm in a non-cooperative competitive market. Specifically, when Firm 2 is using random or static pricing, Firm 1 can increase its revenue by adopting dynamic pricing with demand learning strategy. For example, the average revenue of Firm 1 is 1,436,475 when both companies are using static pricing strategy and noise is 0.2. After changing pricing strategy to dynamic pricing, its average revenue jumps to 6,112,101.

Even if Firm 2 is using dynamic pricing, Firm 1 should also use dynamic pricing with learning to increase revenue. Let us look at the case when Firm 2 is employing dynamic pricing and noise is 0.2. The average revenue of Firm 1 is 100,092 with random pricing or 82,501 with static pricing. However, firm 1 can increase its revenue to 867,265 by employing demand learning based dynamic pricing. This result holds the other way around. That is, when Firm 1's policy is fixed, Firm 2 should use dynamic pricing with demand learning regardless of the competitor's pricing strategy. However, one interesting observation is that the realized revenues decrease significantly if both firms are adopting the learning method compared to the case where both firms are using random pricing or static pricing. It can be interpreted that non-cooperative firms will be worse off as long as there is competition and demand learning.

Next, sensitivity analysis is performed to see how the variation of uncertain parameter affects the revenue of each firm. It can be seen from Figure 2 that, although it's not strictly monotone, the revenue tends to increase as the noise increases. Intuitively, when noise is large, all demand learning methods results come close to the random pricing results, which may provide higher revenue.

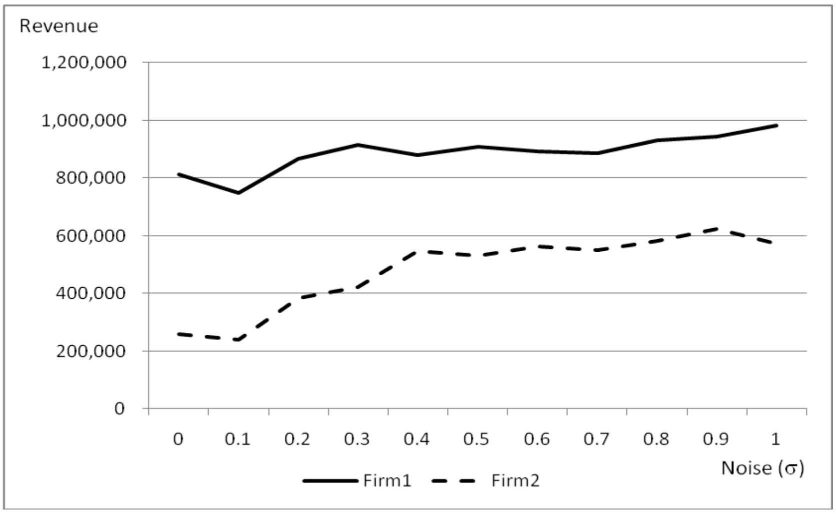


Figure 2. Realized revenues as noise increases

### 4.3 Managerial Implications

Our numerical experiments have several managerial implications. First, they indicate that dynamic pricing together with demand learning is the best strategy to take no matter whether a firm is in a monopoly market or in a non-cooperative competitive market. It is well known that dynamic pricing is an effective way to manipulate market demand and maximize revenue in the short run. However, without an appropriate demand learning strategy, a firm may not be able to forecast customers' response to price changes, which leads to inappropriate dynamic pricing decision and loss of sales opportunity.

Second, the numerical example of a monopoly market demonstrates that a good statistical learning method is crucial to the success of demand learning. Although assumptions of model structure could provide analytical tractability and reduce computational cost, incorrect assumptions about the unobserved dynamics will essentially deteriorate the power of demand learning and result in biased estimation of future demands. On the other hand, the nonparametric FAR model with MCMC algorithms is the-state-of-the-art method for discovering underlying patterns from the data. Despite its sophisticated representation and algorithms, it makes the most of the data that are available, and automatically formulates an equation that best describes the evolution of underlying dynamics of price sensitivity. What is more, the increasing computational power nowadays allows easy and fast implementations of this method.

Third, in a competitive market, a firm’s revenue is determined according to its own pricing decision as well as competitors’ decisions. Similar to the monopoly case, our analysis indicates that a firm can take advantage of demand learning in a competitive market. In practice, it may be difficult to know the demand learning and pricing strategies of competitors, but our results show that it makes sense to employ the proposed demand learning and dynamic pricing strategy even when competitor's information is incomplete.

## 5. CONCLUDING REMARKS

This paper proposed a demand learning strategy from the perspective of evolutionary game theory, and showed how this strategy can be used for the dynamic pricing problem in both monopoly and oligopoly markets. Markov chain Monte Carlo algorithms were developed to estimate unknown parameters and state variables (price sensitivities) in our demand learning model. Nonparametric techniques based on functional-coefficient autoregressive models were incorporated to discover the dynamics of unobserved price sensitivities such that no arbitrary model assumption is needed. After estimating how demand response to price changes, a simulated annealing algorithm and a fixed point algorithm were employed to obtain the optimal pricing policy in a monopoly market and a duopoly market, respectively. The simulation results showed that our new method provides better estimations and predictions of price sensitivity, and is robust over a wide range of underlying state dynamics.

Industrial and market data tends to be messy: the underlying state dynamics could be very complicated and many missing values may exist. Compared with existing demand learning and dynamics pricing methods, our procedure can be directly applied to the data without requiring careful model specification and a great deal of time-consuming data preprocessing. For example, Bertsimas and Perakis [4] assumed a linear function to model demand and price, and Kwon et al. [18] assumed a random walk for describing uncertain parameter. However, our nonparametric demand learning strategy does not make assumptions about model structures, and could adaptively and precisely recover the unobserved price sensitivities. As a result, this learning strategy avoids the risk of model misspecification and is immune to missing values, which are crucial to the following dynamics pricing step. Finally, we provided optimization algorithms that successfully resolve the computational difficulties introduced by the nonparametric demand learning step.

For numerical and theoretical simplicity, our work has focused on homogeneous customers who have the same reference price in mind. The scope of future work could be extended to dynamic pricing problems with heterogeneous customers. Future research could also extend this method to different market scenarios. For example, the efficiency of collaboration between competitors for demand learning can be explored. Moreover, robust optimization approach can be applied to dynamic pricing problems when reliable historical data is unavailable and decision maker can only estimate the boundaries of uncertain parameters.

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## Appendix – DVI Algorithm

The following algorithm is used for solving the non-cooperative competition problem in Section 4.2. When regularity conditions hold, DVI is equivalent to the fixed point problem and it can be solved by an associated fixed point algorithm (see [9] for details of regularity conditions and convergence analysis). The fixed point algorithm based on the iterative scheme is given below:

a) Identify an initial feasible solution $\pi^0 \in \Omega$ and set $k$=0

b) Given $\pi^k$, solve the state dynamics to obtain $D^k$ and $y^k$ by using the demand and price sensitivity dynamics

$$\begin{aligned}
&\frac{dy_i^k}{dt} = \pi_i^{fk} + \sum_{g \in F/f} \pi_i^{gk}, \;\; y_i^k(t_0) = 0, \\
&\frac{dD_i^k}{dt} = \eta_i^f \left(\frac{y_i^k}{|F|(t-t_0)} - \pi_i^{fk}\right), \;\; D_i^{fk}(t_0) = K_{i,0}^f.
\end{aligned}$$

c) Given $\pi^k, D^k, y^k$, solve the adjoint dynamics to obtain $\lambda^k, \sigma^k$.

$$\begin{aligned}
&-\frac{d\lambda_i^{fk}}{dt} = \frac{dH_f}{dD_i^f} = e^{-\rho t}\pi_i^{fk} - \alpha_i^{fk} + \beta_i^{fk} A, \;\; \lambda_i^{fk}(t_f) = 0, \\
&-\frac{d\sigma_i^{fk}}{dt} = \frac{dH_f}{dy_i^f} = \frac{\lambda_i^f \eta_i^f}{|F|(t-t_0)}, \;\; \sigma_i^{fk}(t_f) = 0.
\end{aligned}$$

d) Compute $F_i^{fK}$ using the equation $F_i^{fk} = e^{-\rho t} D_i^{fk} - \lambda_i^{fk} \eta_i^f + \sigma_i^{fk}$

e) Solve the following optimal control problem in order to get an optimal solution ($v^*$**)** for iteration $k$ and call the solution $\pi^{k+1}$.

$$
\begin{aligned}
&\min_{v} J^k(v) = \int_{t_0}^{t_f} \frac{1}{2}(\pi^k - \alpha F^k - v)^T(\pi^k - \alpha F^k - v) \\
&s.t. \\
&v \in \Omega \\
&\frac{dD_i^f}{dt} = \eta_i^f\left(\frac{y_i}{|F|(t-t_0)} - v_i^f\right) \\
&\frac{dy_i}{dt} = v_i^f + \sum_{g \in F \setminus f} v_i^g \\
&A_i^r D_i^f \le C_r^f \\
&D_i^f \ge 0
\end{aligned}
$$

Note that $F_i^{fk}$ computed in step d) is an element of $F^k$.

f) Stopping test. If $\| \pi^{k+1} - \pi^k \| \le \varepsilon$ where $\varepsilon \in R_{++}^1$, stop and optimal price $\pi^* = \pi^{k+1}$, otherwise, set k = k+1 and go to step b.