Dynamic Non-cooperative Games as a Foundation for Modeling Dynamic User Equilibrium*

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Abstract

In this paper we take the point of view that there is a formalism for modeling a within-day dynamic user equilibrium (DUE) that is an extension of traditional differential game theory to accomodate the natural formulation of DUE as an infinite dimensional differential variational inequality (DVI) involving explicit state-dependent time shifts, explicit control variables and explicit equations of state. We also show how a second time scale (day-to-day) may be included to model the learning process behind the formation of demand. An example based on both time scales is included.

Keywords: Dynamic User Equilibrium; Differential Variational Inequality; Optimal Control

1 Introduction

In this paper we take the point of view that there is a formalism for modeling dynamic user equilibrium (DUE) that is not widely understood or applied. That formalism is the extension of traditional differential game theory to accomodate the natural formulation of DUE as an infinite dimensional variational inequality involving explicit state-dependent time shifts, explicit control variables and explicit equations of state. We call this the differential variational inequality (DVI) formalism (Friesz and Mookherjee (2006)). We begin with some foundation material from the theory of deterministic optimal control, and mathematical programming in function spaces. From there we show how time shifts may be considerd by appeal to the notion of G-differentiability. Next we show how dynamic Cournot-Nash-Bertrand games may be formulated as differential variational inequalities, leading to necessary conditions for such dynamic games that are static variational inequalities. We then discuss how functional fixed point algorithms whose subproblems are tractable optimal control problems — without time shifts even when the original dynamic game has time shifts — may be constructed and implemented.

We then show how a well-known DUE model, proposed by Friesz, Bernstein, Smith, Tobin and Wie (1993), may be treated using the apparatus of differential variational inequalities (DVIs). In particular, the DVI formalism is shown to accomodate both path-based and arc-based formulations of DUE, as well as alternative models of delay and explicit queue spill-back constraints. We observe that the DVI formalism allows a direct and quite simple treatment of the first-in-first-out queue discipline. We also observe that the formalism may be extended to account for stochastic phenomena, including both imperfect and incomplete information. We conclude this paper by applying the formalism to create two entirely new formulations of dynamic user equilibrium when: (1) there are dual time scales (day-to-day and within-day); and (2) demand information is uncertain.

2 Differential Variational Inequality with State Dependent Time Shifts

Dynamic systems comprised of game-theoretic agents having control of their own (but not necessarily anyone else's) strategic variables are self-organizing if observable, persistent behavioral patterns and hierarchies emerge with the passage of time. Moreover, time-shifted variational inequalities with explicit state dynamics and explicit controls are

*This paper supplies the mathematical background and a more rigorous theoretical development missing from Friesz and Mookherjee (2006), which is mainly concerned with computation and is largely based on intuitive arguments. The present paper purposely subsumes the previous Friesz and Mookherjee (2006) paper to provide a self-contained reference.

known to arise in the modeling of such systems if the game-theoretic agents have a forward-looking or anticipatory perspective and the emergent behavior is some variety of Cournot-Nash-Bertrand equilibrium, be it static or moving in nature.

Here we take the point of view that infinite dimensional variational inequalities with state dynamics among their constraints and having explicit control variables are direct generalizations of optimal control problems. Because such problems contain ordinary differential equations of state among their constraints, they are one variety of differential variational inequality (DVI) that we refer to as a differential variational inequality with controls (DVIC). It stands to reason that the study of DVICs should involve the derivation of a generalized version of the Pontryagin maximum principal as well as other necessary conditions similar to those encountered in optimal control theory – as we do in Section 2.2. We know of no other manuscripts that use the optimal control perspective taken herein for the study of time-shifted infinite dimensional (dynamic) variational inequalities with state dynamics and explicit controls.

In particular, we will consider the notion of a variational inequality in Hilbert space that includes state dynamics as constraints in the form of a two-point boundary value problem depending parametrically on control variables. Both the principal operator of the variational inequality and the dynamics themselves will involve time shifts that are state-dependent. In fact we consider the following operator

$$x\left(u,u_D,t\right)=\arg\left\{\frac{dx}{dt}=f\left(x,u,u_D,t\right),\ x\left(t_0\right)=x^0,\Gamma\left[x\left(t_f\right),t_f\right]=0\right\}\in\left(\mathcal{H}^1\left[t_0,t_f\right]\right)^n \tag{1}$$

where t_0 and t_f are given and

$$[t_0,t_f]\subseteq\Re^1_+$$

Furthermore $u_D\left(t\right)$ is a shorthand for the shifted control vector

$$u_D\left(t\right)=\begin{pmatrix} u_1\left(t+D_1\left(x_1\right)\right)\\ \vdots \\ u_m\left(t+D_m\left(x_m\right)\right)\end{pmatrix}$$

where $D_i:\left(\mathcal{H}^1\left[t_0,t_f\right]\right)^n\longrightarrow\mathcal{H}^1\left[t_0,t_f\right]$ for each $i\in[1,m]$. The other relevant mappings are

$$\begin{aligned}
f &: \left(\mathcal{H}^1\left[t_0,t_f\right]\right)^n\times\left(L^2\left[t_0,\tau\right]\right)^m\times\left(L^2\left[t_0,t_f\right]\right)^m\times\Re^1_+\longrightarrow\left(L^2\left[t_0,t_f\right]\right)^n\\
\Gamma &: \left(\mathcal{H}^1\left[t_0,t_f\right]\right)^n\times\Re^1_+\longrightarrow\left(\mathcal{H}^1\left[t_0,t_f\right]\right)^r\\
u &\in U\subseteq\left(L^2\left[t_0,t_f\right]\right)^m;\ u_D:\left(\mathcal{H}^1\left[t_0,t_f\right]\right)^n\times\Re^1_+\longrightarrow\left(L^2\left[t_0,t_1\right]\right)^m
\end{aligned}$$

where

$$t_1=t_f+\max\left\{D_i\left[x\left(t_f\right)\right]:i\in[1,m]\right\} \tag{2}$$

In the above $\left(L^2\left[t_0,t_f\right]\right)^m$ is the m-fold product of the space of square-integrable functions $L^2\left[t_0,t_f\right]$, while $\left(\mathcal{H}^1\left[t_0,t_f\right]\right)^n$is the n-fold product of the Sobolev space $\mathcal{H}^1\left[t_0,t_f\right]$.

Additionally we invoke the following regularity condition for the two-point boundary value problem (1):

Definition 1 *Regular Dynamics. We shall say the state dynamics operator $x\left(u,u_D,t\right)$ given by (1) is $\left(x^0,U,\Gamma\right)$-regular if the terminal state constraint $\Gamma\left[x\left(t_f\right),t_f\right]=0$ is reachable from the given initial state x^0 for all $u\in U$.*

The notation $x\left(u,u_D,t\right)$ is a direct generalization of that used by Minoux (1986) to describe how the Pontryagin minimum principle of optimal control theory may be derived using notions from infinite dimensional mathematical programming; it denotes an operator which determines the state vector for any pair of shifted and un-shifted control vectors. In order to use the operator notation $x\left(u,u_D,t\right)$, we will invoke $\left(x^0,U,\Gamma\right)$-regularity to ensure that the parametric boundary value problem (1) is well posed. Such a regularity condition should not be interpreted as an *a priori* stipulation that the variational inequality to be introduced below has a solution; rather it is a stipulation that the constrained dynamics of (1) have a solution for all controls that are considered pertinent to the problem of interest.

2.1 A Related Optimal Control Problem

Before studying differential variational inequalities with state-dependent time shifts, we need to derive necessary conditions for a related optimal control problem. That derivation relies on the notion of a so-called *G-derivative*:

Definition 2 *(G-differentiable, Minoux (1986)) Let V be a normed vector space and J be a functional on V. If for all $\varphi \in V$ the limit $\delta J(v,\varphi)$ defined by*

$$\delta J(v,\varphi) \equiv \lim_{\theta \to 0} \frac{J(v+\theta\varphi) - J(v)}{\theta}$$

exists, then J is said to be differentiable in the sense of Gateaux (G-differentiable) at $v \in V$.

With the preceding background, we consider the following optimal control problem:

$$\min \Gamma\left[x(t_f), t_f\right] + \int_{t_0}^{t_f} G(x,u,u_D,t)dt \tag{3}$$

subject to

$$\frac{dx}{dt} = f(x,u,u_D,t)\,;\; x(t_0) = x^0 \tag{4}$$

$$u \in U \tag{5}$$

This is a non-standard optimal control problem, and we will need its necessary conditions. In fact we will state and prove the following result:

Theorem 3 *(Necessary Conditions for Optimal Control with State-Dependent Time Shifts) If (i) $u \in U \subseteq \left(L^2[t_0,\tau]\right)^m$; (ii) $u_D \in \left(L^2[t_0,t_f]\right)^m$; (iii) the operator $x(u,u_D,t) : \left(L^2[t_0,t_f]\right)^m \times \left(L^2[t_0,\tau]\right)^m \longrightarrow \left({\mathcal{H}_\infty}^1[t_0,t_f]\right)^n$ is $\left(x^0,U,\Gamma\right)$-regular, continuous and G-differentiable with respect to u and u_D; (iv) $D_i(x_i) : \left(\mathcal{H}^1[t_0,t_f]\right)^n \longrightarrow \mathcal{H}^1[t_0,t_f]$ is continuously differentiable with respect to x_i for each $i \in [1,m]$; (v) $\Gamma[x,t] : \left(\mathcal{H}^1[t_0,t_f]\right)^n \times \Re^1_+ \longrightarrow \mathcal{H}^1[t_0,t_f]$ is continuously differentiable with respect to x; (vi) $G(x,u,u_D,t) : \left(\mathcal{H}^1[t_0,t_f]\right)^n \times \left(L^2[t_0,\tau]\right)^m \times \left(L^2[t_0,\tau]\right)^m \times \Re^1_+ \longrightarrow L^2[t_0,t_f]$ is continuously differentiable with respect to x, u and u_D; (vii) $f(x,u,u_D,t) : \left(\mathcal{H}^1[t_0,t_f]\right)^n \times \left(L^2[t_0,\tau]\right)^m \times \left(L^2[t_0,\tau]\right)^m \times \Re^1_+ \longrightarrow \left(L^2[t_0,t_f]\right)^n$ is continuously differentiable with respect to x, u and u_D; (viii) $U \subseteq \left(L^2[t_0,\tau]\right)^m$ is convex and compact; and (ix) $x^0 \in \Re^n$*

then any solution $u^* \in U$ of the optimal control problem (3) through (5) obeys the following necessary conditions:

1. *the finite dimensional variational inequality principle:*

$$\sum_{i=1}^{m} \frac{\partial H_1^*}{\partial u_i}\left(u_i - u_i^*\right) \geq 0 \quad \forall t \in \left[t_0, D_i(x(t_0))\right], u \in U$$

$$\sum_{i=1}^{m} \left\{ \frac{\partial H_1^*}{\partial u_i} + \left[\frac{\partial H_1^*}{\partial (u_D)_i} \frac{1}{1+\sum_{j=1}^{m} \frac{\partial D_i(x^*)}{\partial x_j}\frac{dx_j^*}{dt}} \right]_{s_i(t)} \right\} \left(u_i - u_i^*\right) \geq 0 \;\; \forall t \in \left[D_i(x^*(t_0)), t_f + D_i(x^*(t_f))\right], u \in U$$

where $s_i(t) = \arg\left[s = t - D_i(x(s))\right] \;\; \forall t \in \left[D_i(x^*(t_0)), t_f + D_i(x^*(t_f))\right], i \in [1,m]$ and $H_1^* = H_1(x^*,u^*,u_D^*,\lambda^*,t) = G(x^*,u^*,u_D^*,t) + (\lambda^*)^T f(x^*,u^*,u_D^*,t) \;\; \forall t \in [t_0,t_f]$;

2. *the state dynamics*

$$\frac{dx^*}{dt} = f(x^*,u^*,u_D^*,t)\,;\; x^*(t_0) = x^0; \quad \text{and}$$

3. *the adjoint dynamics*

$$(-1)\frac{d\lambda^*}{dt} = \nabla_x (\lambda^*)^T f(x^*,u^*,u_D^*,t)\,;\; \lambda^*(t_f) = \frac{\partial \Gamma\left[x^*(t_f), t_f\right]}{\partial x}.$$

Proof. The below proof extends the fixed time shift analysis of **?** to state-dependent time shifts. Note that

$$x(u,u_D,t) = x(t_0) + \int_{t_0}^{t} f[x(u,u_D,t),u,u_D,t]\,dt$$

It is immediate that

$$x(u+\theta\rho,u_D+\theta\rho_D) = x(t_0) + \int_{t_0}^{t} f[x(u+\theta\rho,u_D+\theta\rho_D),u+\theta\rho,u_D+\theta\rho_D,t]\,dt$$

Consequently,

$$\begin{aligned}\delta x(u,\rho;u_D,\rho_D) &= \int_{t_0}^{t}\left\{\frac{\partial f[x(u,u_D,t),u,u_D,t]}{\partial x}\delta x(u,\rho;u_D,\rho_D) + \frac{\partial f[x(u),u,u_D,t]}{\partial u}\delta u(\rho)\right.\\ &\quad\left.+\frac{\partial f[x(u),u,u_D,t]}{\partial u_D}\delta u_D(\rho_D)\right\}dt\end{aligned}$$

where the G-derivatives of u and u_D obey

$$\delta u(\rho) = \lim_{\theta\longrightarrow 0}\frac{(u+\theta\rho)-u}{\theta} = \rho;\;\; \delta u_D(\rho_D) = \lim_{\theta\longrightarrow 0}\frac{(u_D+\theta\rho_D)-u_D}{\theta} = \rho_D$$

Employing the shorthand $y = \delta x(u,\rho;u_D,\rho_D)$, we have the integral equation

$$y = \int_{t_0}^{t}\left[\frac{\partial f}{\partial x}y + \frac{\partial f}{\partial u}\rho + \frac{\partial f}{\partial u_D}\rho_D\right]dt \tag{6}$$

It is of course immediate from this integral equation that y obeys

$$\frac{dy}{dt} = \frac{\partial f}{\partial x}y + \frac{\partial f}{\partial u}\rho + \frac{\partial f}{\partial u_D}\rho_D;\quad y(t_0) = 0 \tag{7}$$

which is recognized as an initial value problem, verifying that the G-derivative of x is well defined. The G-derivative of J obeys

$$\begin{aligned}\delta J(u,\rho;u_D,\rho_D) &= \left[\frac{\partial\Gamma[x(t),t]}{\partial x}\delta x(u,\rho;u_D,\rho_D)\right]_{t_0}^{t_f} + \int_{t_0}^{t_f}\left[\frac{\partial G}{\partial x}\delta x(u,\rho;u_D,\rho_D) + \frac{\partial G}{\partial u}\delta u(\rho) + \frac{\partial G}{\partial u_D}\delta u(\rho_D)\right]\\ &= \frac{\partial\Gamma[x(t_f),t_f]}{\partial x}y(t_f) + \int_{t_0}^{t_f}\left[\frac{\partial G}{\partial x}y + \frac{\partial G}{\partial u}\rho + \frac{\partial G}{\partial u_D}\rho_D\right]dt\end{aligned}$$

We introduce *adjoint variables* λ defined by the final value problem

$$-\frac{d\lambda}{dt} = \left(\frac{\partial f}{\partial x}\right)^T\lambda + \left(\frac{\partial G}{\partial x}\right)^T;\quad \lambda(t_f) = \frac{\partial\Gamma[x(t_f),t_f]}{\partial x} \tag{8}$$

so that

$$\delta J(u,\rho;u_D,\rho_D) = \int_{t_0}^{t_f}\left[-\left(\frac{d\lambda}{dt}\right)^T y - \lambda^T\frac{\partial f}{\partial x}y + \frac{\partial G}{\partial u}\rho + \frac{\partial G}{\partial u_D}\rho_D\right]dt \tag{9}$$

Note that

$$\begin{aligned}\left[\lambda^T y\right]_{t_0}^{t_f} &= [\lambda(t_f)]^T y(t_f) - [\lambda(t_0)]^T y(t_0)\\ &= \frac{\partial\Gamma[x(t_f),t_f]}{\partial x}y(t_f)\end{aligned}$$

due to (8) and the fact that $y(t_0) = 0$, so an integration by parts yields

$$\begin{aligned} \int_{t_0}^{t_f} -\left(\frac{d\lambda}{dt}\right)^T y dt &= \int_{t_0}^{t_f} \lambda^T \frac{dy}{dt} dt - \left[\lambda^T y\right]_{t_0}^{t_f} \\ &= \int_{t_0}^{t_f} \lambda^T \frac{dy}{dt} dt - \frac{\partial \Gamma [x(t_f), t_f]}{\partial x} y(t_f) \\ &= \int_{t_0}^{t_f} \lambda^T \left[\frac{\partial f}{\partial x} \cdot y + \frac{\partial f}{\partial u} \cdot \rho + \frac{\partial f}{\partial u_D} \rho_D\right] dt - \frac{\partial \Gamma [x(t_f), t_f]}{\partial x} y(t_f) \end{aligned} \tag{10}$$

It follows that

$$\begin{aligned} \delta J(u, \rho; u_D, \rho_D) &= \frac{\partial \Gamma [x(t_f), t_f]}{\partial x} y(t_f) + \int_{t_0}^{t_f} \left\{\lambda^T \left[\frac{\partial f}{\partial x} \cdot y + \frac{\partial f}{\partial u} \cdot \rho + \frac{\partial f}{\partial u_D} \rho_D\right]\right. \\ &\qquad \left. -\lambda^T \frac{\partial f}{\partial x} y + \frac{\partial G}{\partial u} \rho + \frac{\partial G}{\partial u_D} \rho_D\right\} dt - \frac{\partial \Gamma [x(t_f), t_f]}{\partial x} y(t_f) \\ &= \int_{t_0}^{t_f} \left[\lambda^T \frac{\partial f}{\partial u} + \frac{\partial G}{\partial u}\right] \rho dt + \int_{t_0}^{t_f} \left[\lambda^T \frac{\partial f}{\partial u_D} + \frac{\partial G}{\partial u_D}\right] \rho_D dt \end{aligned}$$

Defining $H_1(x, u, u_D, \lambda, t) = G(x, u, u_D, t) + \lambda^T f(x, u, u_D, t)$, we have

$$\delta J(u, \rho; u_D, \rho_D) = \int_{t_0}^{t_f} \left[\frac{\partial H_1}{\partial u} \rho + \frac{\partial H_1}{\partial u_D} \rho_D\right] dt \tag{11}$$

as an expression for the G-derivative of the criterion with respect to both u and u_D. Moreover, terms of the form

$$\int_{t_0}^{t_f} \frac{\partial H_1}{\partial (u_D)_i} (\rho_D)_i \, dt = \int_{t_0}^{t_f} \frac{\partial H_1}{\partial (u_D)_i} \delta u_i (t + D_i(x_i)) \, dt$$

may be re-expressed by making the change of variables

$$\Delta_i = t + D_i(x(t)) \Longleftrightarrow t = \Delta_i - D_i(x(t))$$

Because the $D_i(x)$ are differentiable with respect to x_i, the implicit function theorem gives

$$\frac{dt}{d\Delta_i} = -\frac{\partial [t - \Delta_i + D_i(x)] / \partial \Delta_i}{\partial [t - \Delta_i + D_i(x)] / \partial t} = \frac{1}{1 + \sum_{j=1}^m \frac{\partial D_i(x)}{\partial x_j} \dot{x_j}}$$

or,

$$dt = \frac{1}{1 + \sum_{j=1}^m \frac{\partial D_i(x)}{\partial x_j} \dot{x_j}} d\Delta_i \tag{12}$$

Note that

$$t = t_0 \Longrightarrow \Delta_i = t_0 + D_i(x(t_0)) \, ; \text{ Putting } t = t_f \Longrightarrow \Delta_i = t_f + D_i(x(t_f))$$

Furthermore, without loss of generality, we may take $\delta (u_D)_i = 0$ for any time $t < D_i(x(t_0))$ and $\delta (u)_i = 0$ for any time $t > D_i(x(t_0))$. A change of variables based on (12) leads to

$$\begin{aligned} \int_{t_0}^{t_f} \frac{\partial H_1}{\partial (u_D)_i} (\rho_D)_i \, dt &= \int_{D_i(x_i(t_0))}^{t_f + D_i(x_i(t_f))} \frac{\partial H_1}{\partial (u_D)_i} \delta (u_D)_i \, dt \\ &= \int_{D_i(x_i(t_0))}^{t_f + D_i(x_i(t_f))} \left[\frac{\partial H_1}{\partial (u_D)_i} \frac{1}{1 + \sum_{j=1}^m \frac{\partial D_i(x)}{\partial x_j} \dot{x_j}}\right]_{s_i(t)} \delta (u)_i \, dt \\ &= \int_{D_i(x_i(t_0))}^{t_f + D_i(x_i(t_f))} \left[\frac{\partial H_1}{\partial (u_D)_i} \frac{1}{1 + \sum_{j=1}^m \frac{\partial D_i(x)}{\partial x_j} \dot{x_j}}\right]_{s_i(t)} \rho_i dt \end{aligned} \tag{13}$$

where $s_i(t)$ obeys $s_i(t) = \arg[s = t - D_i(x(s))]$ for any given instant of time t at which the term

$$\frac{\partial H_1}{\partial (u_D)_i} \frac{1}{1+\sum_{j=1}^m \frac{\partial D_i(x)}{\partial x_j} \dot{x_j}}$$

must be evaluated. Note that the change of variables in (13) has re-expressed the G-derivative of u_D as a derivative of u. We next note that

$$\int_{t_0}^{t_f} \frac{\partial H_1}{\partial u_i}\rho_i dt = \int_{t_0}^{D_i(x_i(t_0))} \frac{\partial H_1}{\partial u_i}\rho_i dt + \int_{D_i(x_i(t_0))}^{t_f+D_i(x_i(t_f))} \frac{\partial H_1}{\partial u_i}\rho_i dt \tag{14}$$

This last result means that for the change of variables introduced above the G-derivative is expressible in terms of ρ; that is

$$\delta J(u,\rho;u_D,\rho_D) = \delta J(u,\rho;u,\rho) \equiv \delta J(u,\rho)$$

Using (13) and (14) we obtain

$$[\delta J(u,\rho)]_i = \int_{t_0}^{D_i(x(t_0))} \frac{\partial H_1}{\partial u_i}\rho_i dt + \int_{D_i(x(t_0))}^{t_f+D_i(x(t_f))} \left\{ \frac{\partial H_1}{\partial u_i} + \left[\frac{\partial H_1}{\partial (u_D)_i} \frac{1}{1+\sum_{j=1}^m \frac{\partial D_i(x)}{\partial x_j}\dot{x_j}} \right]_{t=s_i} \right\} \rho_i dt$$

Note that in the above each component of $\delta J(u,\rho)$ has a different upper limit of integration and thereby we cannot give an inner product representation of the G-derivative in terms of a gradiemt and a direction vector. However, without loss of generality we may define $\delta(u)_i = 0$ for any $t > t_f + D_i(x(t_f))$. Since $\delta(u)_i = \rho_i$ we may write

$$[\delta J(u,\rho)]_i = \int_{t_0}^{D_i(x(t_0))} \frac{\partial H_1}{\partial u_i}\rho_i dt + \int_{D_i(x(t_0))}^{t_1} \left\{ \frac{\partial H_1}{\partial u_i} + \left[\frac{\partial H_1}{\partial (u_D)_i} \frac{1}{1+\sum_{j=1}^m \frac{\partial D_i(x)}{\partial x_j}\dot{x_j}} \right]_{t=s_i} \right\} \rho_i dt$$

where t_1 is defined by (2) and the same for all $i \in [1,m]$, which has the effect of defining the G-derivative of the criterion as

$$\delta J u,\rho) = \int_{t_0}^{t_1} \left[\frac{\partial H_1}{\partial u}\rho\right] dt$$

Optimality requires $u^* \in U$ to obey

$$\delta J(u^*,\rho) \geq 0 \quad \forall \rho \geq 0 \tag{15}$$

which directly yields the desired necessary conditions when it is observed that each direction may be stated as $\rho = (u - u^*)$ for some $u \in U$. ■

The following result, stemming directly from the above proof, is also important:

Corollary 4 *(Gradient of the Criterion in the Presence of Time Shifts) For regularity in the sense of Definition 5, the gradient of the criterion (3) is defined by*

$$[\nabla J(u)]_i = \begin{cases} \frac{\partial H_1}{\partial u_i} & \text{if } t \in [t_0, D_i(x_i(t_0))] \\ \frac{\partial H_1}{\partial u_i} + \left[\frac{\partial H_1}{\partial (u_D)_i} \frac{1}{1+\sum_{j=1}^m \frac{\partial D_i(x)}{\partial x_j}\dot{x_j}} \right]_{s_i(t)} & \text{if } t \in [D_i(x^*(t_0)), t_f + D_i(x^*(t_f))] \end{cases}$$

for $i = [1,m]$.

Proof. By the Riesz representation theorem we know

$$\delta J(u^*,\rho) = \langle \nabla J(u^*), (u - u^*) \rangle \quad \forall u \in U \tag{16}$$

The result is then immediate. ■

2.2 Statement of a DVI with State Dependent Time Shifts

With the above background we are now ready to study the following problem:

$$\text{find } u^* \in U \text{ such that}$$

$$\langle F\left(x\left(u^*, u_D^*\right), u^*, u_D^*, t\right), u - u^* \rangle \geq 0 \text{ for all } u \in U \tag{17}$$

where

$$x\left(u, u_D, t\right) = \arg\left\{\frac{dx}{dt} = f\left(x, u, u_D, t\right),\ x\left(t_0\right) = x^0, u \in U, \Gamma\left[x\left(t_f\right), t_f\right] = 0\right\} \in \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \tag{18}$$

We refer to (17) as a differential variational inequality with explicit controls and time shifts, abbreviated $DVIC(F, f, \Gamma, D, U, x^0)$.

2.2.1 Necessary Conditions

To develop necessary conditions for solutions of (17) we will rely on the following notion of regularity:

Definition 5 *[Regularity of $DVIC(F, f, \Gamma, D, U, x^0)$] We call $DVIC(F, f, \Gamma, D, U, x^0)$ regular if: (i) $u \in U \subseteq \left(L^2\left[t_0, \tau\right]\right)^m$; (ii) $u_D \in \left(t_f^2\left[t_0, t_f\right]\right)^m$; (iii) the operator $x\left(u, u_D, t\right) : \left(L^2\left[t_0, t_f\right]\right)^m \times \left(L^2\left[t_0, \tau\right]\right)^m \longrightarrow \left(\mathcal{H}_\infty{}^1\left[t_0, t_f\right]\right)^n$ is $\left(x^0, U, \Gamma\right)$-regular, continuous and G-differentiable with respect to u and u_D; (iv) $D_i\left(x\right) : \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \longrightarrow \mathcal{H}^1\left[t_0, t_f\right]$ is continuously differentiable with respect to x_i, for each $i \in [1, m]$; (v) $\Gamma\left(x, t\right) : \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \times \Re_+^1 \longrightarrow \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^r$ is continuously differentiable with respect to x; (vi) $F(x, u, u_D, t) : \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \times \left(L^2\left[t_0, \tau\right]\right)^m \times \left(L^2\left[t_0, t_f\right]\right)^m \times \Re_+^1 \longrightarrow \left(L^2\left[t_0, t_f\right]\right)^m$ is continuous with respect to x and u; (vii) $f\left(x, u, u_D, t\right) : \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \times \left(L^2\left[t_0, \tau\right]\right)^m \times \left(L^2\left[t_0, t_f\right]\right)^m \times \Re_+^1 \longrightarrow \left(L^2\left[t_0, t_f\right]\right)^n$ is continuously differentiable with respect to x, u and u_D; (viii) $U \subseteq \left(L^2\left[t_0, \tau\right]\right)^m$ is convex and compact; and (ix) $x^0 \in \Re^n$.*

We next note that (17) may be restated as the following optimal control problem

$$\min \gamma^T \Gamma\left[x\left(t_f\right), t_f\right] + \int_{t_0}^{t_f} \left[F\left(x^*, u^*, u_D^*, t\right)\right]^T u dt \tag{19}$$

subject to

$$\frac{dx}{dt} = f\left(x, u, u_D, t\right);\ x\left(t_0\right) = x^0 \tag{20}$$

$$u \in U \tag{21}$$

where $x^* = x\left(u^*, u_D^*\right)$ is the optimal state vector and $\gamma \in \Re^r$ is the vector of dual variables for the terminal constraints $\Gamma\left[x\left(t_f\right), t_f\right] = 0$. We point out that this optimal control problem is a mathematical abstraction and of no use for computation, since its criterion depends on knowledge of the variational inequality solution u^*. In what follows we will need the Hamiltonian for (19) through (21), namely

$$H_2\left(x, u, u_D, \lambda, t\right) = \left[F\left(x^*, u^*, u_D^*, t\right)\right]^T u + \lambda^T f\left(x, u, u_D, t\right) \tag{22}$$

where $\lambda\left(t\right)$ is the adjoint vector that solves the adjoint equations and transversality conditions for given state variables and controls. It is now a relatively easy matter to derive the necessary conditions stated in the following theorem:

Theorem 6 *[Necessary Conditions for $DVIC(F, f, \Gamma, D, U, x^0)$] When regularity in the sense of Definition 5 holds, solutions $u^* \in U$ of $DVIC(F, f, \Gamma, D, U, x^0)$ must obey:*

1. *the finite dimensional variational inequality principle:*

$$\sum_{i=1}^{m} \left[F_i\left(x^*, u^*, u_D^*, t\right) + \sum_{j=1}^{m} \lambda_j \frac{\partial f_i\left(x^*, u^*, u_D^*, t\right)}{\partial u_i}\right]\left(u_i - u_i^*\right) \geq 0 \quad \forall t \in \left[t_0, D_i\left(x\left(t_0\right)\right)\right], u \in U$$

$$\sum_{i=1}^{m}\left\{F_i\left(x^*,u^*,u_D^*,t\right)+\sum_{j=1}^{m}\lambda_j\frac{\partial f_j\left(x^*,u^*,u_D^*,t\right)}{\partial u_i}\right.$$
$$\left.+\left[\lambda_j\frac{\partial f_j\left(x^*,u^*,u_D^*,t\right)}{\partial\left(u_D\right)_i}\frac{1}{1+\sum_{j=1}^{m}\frac{\partial D_i(x^*)}{\partial x_j}f_j\left(x^*,u^*,u_D^*,t\right)}\right]_{s_i(t)}\right\}\left(u_i-u_i^*\right)\geq 0$$
$$\forall t\in\left[D_i\left(x^*\left(t_0\right)\right),t_f+D_i\left(x^*\left(t_f\right)\right)\right],u\in U$$

2. the state dynamics

$$\frac{dx^*}{dt}=f\left(x^*,u^*,u_D^*,t\right);\ x^*\left(t_0\right)=x^0;\quad \textit{and}$$

3. the adjoint dynamics

$$(-1)\frac{d\lambda^*}{dt}=\nabla_x\left(\lambda^*\right)^T f\left(x^*,u^*,u_D^*,t\right);\ \lambda^*\left(t_f\right)=\nu^T\frac{\partial\Gamma\left[x^*\left(t_f\right),t_f\right]}{\partial x}$$

where $\nu\in\Re^r$ is the vector of dual variables for the terminal constraints $\Gamma\left[x\left(t_f\right),t_f\right]=0$.

Proof. $DVIC(F,f,\Gamma,D,U,x^0)$ is equivalent to the optimal control problem

$$\min\nu^T\Gamma\left[x\left(t_f\right),t_f\right]+\int_{t_0}^{t_f}\left[F\left(x^*,u^*,u_D^*,t\right)\right]^T u dt$$

subject to

$$\begin{aligned}\frac{dx}{dt}&=f\left(x,u,u_D,t\right);\quad x\left(t_0\right)=x^0\\ u&\in U\end{aligned}$$

with Hamiltonian $H_2\left(x,u,u_D,\lambda,t\right)=\left[F\left(x^*,u^*,u_D^*,t\right)\right]^T u+\lambda^T f\left(x,u,u_D,t\right)$ .By virtue of regularity we may apply Theorem 3; the necessary conditions follow immediately. ■

2.3 Fixed Point Formulation and Algorithm

Furthermore, there is a fixed point form of $DVIC(F,f,\Gamma,D,U,x^0)$. In particular we state and prove the following result:

Theorem 7 *(fixed point formulation of $DVIC(F,f,\Gamma,D,U,x^0)$) When regularity in the sense of Definition 5 holds and $f\left(x,u,u_D,t\right):\left(\mathcal{H}^1\left[t_0,t_f\right]\right)^n\times\left(L^2\left[t_0,\tau\right]\right)^m\times\left(L^2\left[t_0,t_f\right]\right)^m\times\Re^1_+\longrightarrow\left(L^2\left[t_0,t_f\right]\right)^n$ is convex, $DVIC(F,f,\Gamma,D,U,x^0)$ is equivalent to the following fixed point problem:*

$$u=P_U\left[u-\alpha F\left(x\left(u,u_D,t\right),u,u_D,t\right)\right]$$

where $P_U\left[.\right]$ is the minimum norm projection onto $U\subseteq\left(L^2\left[t_0,\tau\right]\right)^m$ and $\alpha\in\Re^1_{++}$.

Proof. The fixed point problem considered requires that

$$u=\arg\min_v\left\{\frac{1}{2}\left\|u-\alpha F\left(x\left(u,u_D,t\right),u,u_D,t\right)-v\right\|^2:v\in U\right\}\tag{23}$$

where $\alpha\in\Re^1_{++}$ is any strictly positive real number. That is, we seek the solution of the optimal control problem

$$\min_v\gamma^T\Gamma\left[x\left(t_f\right),t_f\right]+\int_{t_0}^{t_f}\frac{1}{2}\left[u-\alpha F\left(x,u,u_D,t\right)-v\right]^2dt$$

subject to

$$\begin{aligned} \frac{dx}{dt} &= f(x, v, v_D, t)\,;\ x(t_0) = x^0 \\ u &\in U \end{aligned}$$

where u and u_D are treated as fixed vectors. Because of regularity and the assumed convexity of $f(x, v, v_D, t)$, a necessary and sufficient condition for a solution $v^* \in U$ of this optimal control problem is

$$[\nabla_v H_3 (x^*, v^*, v_D^*, \eta^*, t)]^T (v - v^*) \geq 0 \qquad \forall v \in U \tag{24}$$

where $H_3 (x, v, v_D, \eta, t) = \frac{1}{2} [u - \alpha F (x, u, u_D, t) - v]^2 + \eta^T f (x, v, v_D, t)$ and for given x and v

$$\eta = \arg \left\{ (-1) \frac{d\eta}{dt} = \nabla_x H_3 (x, v, v_D, \eta, t)\,,\ \eta (t_f) = \gamma^T \frac{\partial \Gamma [x (t_f), t_f]}{\partial x (t_f)} \right\}$$

Note that $\nabla_v H_3 (x, v, v_D, \eta, t) = -u + \alpha F (x, u, u_D, t) + v + \nabla_v \eta^T f (x, v, v_D, t)$. Because $u = v$ by virtue of (23) we have

$$\nabla_u H_3 (x, v, v_D, \eta, t) = \alpha F (x, u, u_D, t) + \nabla_u \eta^T f (x, u, u_D, t) \tag{25}$$

Now if we set $\lambda = \frac{\eta}{\alpha}$; we have

$$\left[F (x^*, u^*, u_D^*, t) + \nabla_u (\lambda^*)^T f (x^*, u^*, t) \right]^T (u - u^*) \geq 0 \qquad \forall v \in U$$

which is identical to the finite dimensional variational inequality principle of Theorem 6. The other optimality conditions are also identical. This completes the proof. ■

Naturally there is an associated fixed point algorithm based on the iterative scheme

$$u^{k+1} = P_U \left[u^k - \alpha F \left(x \left(u^k, u_D^k \right), u^k, u_D^k, t \right) \right]$$

The detailed structure of the fixed point algorithm is:
Step 0. Initialization: identify an initial feasible solution $u^0 \in U$ and set $k = 0$.
Step 1. Solve optimal control problem: call the solution of the following optimal control problem u^{k+1}.

$$\min_v J^k (v) = \gamma^T \Gamma [x (t_f), t_f] + \int_{t_0}^{t_f} \frac{1}{2} \left[u^k - \alpha F \left(x^k, u^k, u_D^k, t \right) - v \right]^2 dt \tag{26}$$

$$\text{subject to } \frac{dx}{dt} = f (x, v, v_D, t)\,;\ x (t_0) = x^0 \tag{27}$$

$$v \in U \tag{28}$$

Step 2. Stopping test: if $\left\| u^{k+1} - u^k \right\| \leq \varepsilon$ where $\varepsilon \in \Re^1_{++}$ is a preset tolerance, stop and declare $u^* \approx u^{k+1}$. Otherwise set $k = k + 1$ and and go to Step 1.
The convergence of this algorithm is guaranteed by the following result:

Theorem 8 *When $DVIC(F, f, \Gamma, D, U, x^0)$ is regular in the sense of Definition 5 and $f (x, u, u_D, t) : \left(\mathcal{H}^1 [t_0, t_f] \right)^n \times \left(L^2 [t_0, \tau] \right)^m \times \left(L^2 [t_0, t_f] \right)^m \times \Re^1_+ \longrightarrow \left(L^2 [t_0, t_f] \right)^n$ is convex, while additionally $F (x, u, u_D, t)$ is strongly monotonic for $u \in U$, the fixed point algorithm presented above converges.*

Proof. Consider

$$u^{k+1} - u^* = P_U \left[u^k - \alpha F \left(x \left(u^k, u_D^k \right), u^k, u_D^k, t \right) \right] - P_U \left[u^* - \alpha F \left(x \left(u^*, u_D^* \right), u^*, u_D^*, t \right) \right]$$

and note that P_U is a contraction; that is, the projection of a vector is never greater in length than the length of the vector itself. Thus

$$\| P_U (v) \| \leq \| v \|$$

for any $v \in U \subseteq \left(L^2 [t_0, \tau] \right)^m$. Define

$$F^k = F \left(x \left(u^k, u_D^k \right), u^k, u_D^k, t \right);\ F^* = F \left(x \left(u^*, u_D^* \right), u^*, u_D^*, t \right)$$

Because F obeys a strong monotonicity condition, we have

$$\langle F^k - F^*, u^k - u^* \rangle \geq \varepsilon \left\| u^k - u^* \right\|$$

where $\varepsilon \in \Re^1_{++}$. We also know that both $\left\| F^k - F^* \right\|$ and $\left\| u^k - u^* \right\|$ are bounded, by virtue of the boundedness of U and the continuity of F. Consequently, there must exist $\beta \in \Re^1_{++}$ such that

$$\left\| F^k - F^* \right\|^2 \leq \beta \left\| u^k - u^* \right\|^2 \tag{29}$$

The contractive property of P_U and the strong monotonicity of F together with property (29) mean

$$\begin{aligned} \left\| u^{k+1} - u^* \right\|^2 &\leq \left\| \left(u^k - u^*\right) - \alpha \left(F^k - F^*\right) \right\|^2 \\ &= \left\| u^k - u^* \right\|^2 + \alpha^2 \left\| F^k - F^* \right\|^2 - 2\alpha \langle F^k - F^*, u^k - u^* \rangle \\ &\leq (1 + \beta - 2\alpha\varepsilon) \left\| u^k - u^* \right\|^2 \end{aligned}$$

Note that we may chose $\alpha > 0$ such that $1 + \beta - 2\alpha\varepsilon < 1$ which is equivalent to $\alpha > \frac{\beta}{2\varepsilon}$ a condition ensuring

$$\left\| u^{k+1} - u^* \right\|^2 < \left\| u^k - u^* \right\|^2$$

Consequently, the algorithm is a strict contraction mapping and convergence is assured. ■

2.4 Descent in Hilbert Space for the Projection Sub-Problems

It is important to realize that the fixed point algorithm of Section 2.3 can be carried out in continuous time provided we employ a continuous time representation of the solution of each subproblem (26)-(28) from Step 1 of the fixed point algorithm. This may be done using a continuous time gradient projection method. For our present circumstances, that algorithm may be stated as

Descent Algorithm in Hilbert Space for the Projection Sub-Problems

Step 0. *Initialization.* Pick $v^{k,0}(t) \in U$ and set $j = 0$.
Step 1. *Finding state variables. Solve the state dynamics*

$$\frac{dx}{dt} = f\left(x, v^{k,j}, v_D^{k,j}, t\right) \tag{30}$$

$$x(t_0) = x^0 \tag{31}$$

Call the solution $x^{k,j}(t)$.In the event a discrete time method is used to solve the state dynamics (30) and (31), curve fitting is used to obtain the continuous time state vector $x^{k,j}(t)$.
Step 2. *Finding adjoint variables.* Solve the adjoint dynamics

$$(-1)\frac{d\lambda}{dt} = \nabla_x H^k \left|_{x=x^{k,j}} \right. ; \ \lambda(t_f) = \frac{\partial \Gamma\left[x^{k,j}(t_f), t_f\right]}{\partial x(t_f)} \tag{32}$$

where

$$H^k = \frac{1}{2}\left[u^k - \alpha F\left(x^k, u^k, u_D^k, t\right) - v\right]^2 + \lambda^T f\left(x, v^{k,j}, v_D^{k,j}, t\right)$$

Call the solution $\lambda^{k,j}(t)$.In the event a discrete time method is used to solve the adjoint dynamics (**??**) and (32), curve fitting is used to obtain the continuous time adjoint vector $\lambda^{k,j}(t)$.
Step 3. *Finding the gradient.* Determine

$$\nabla_v J^{k,j}(t) = \nabla_v H^k$$

Step 4. *Stopping test. For a fixed and suitably small fixed step size*

$$\theta_k \in \Re^1_{++}$$

determine

$$v^{k,j+1}(t) = P_U\left[v^{k,j}(t) - \theta_k \nabla_v J^{k,j}\right] \tag{33}$$

In the event a discrete time method is used to solve the above projection subproblem, curve fitting is used to obtain the continuous time control vector (33).

Step 5. *Stopping test.* For $\varepsilon_2 \in \Re^1_{++}$, a pre-set tolerance, stop if $\left\|v^{k,j+1} - v^{k,j}\right\| < \varepsilon_1$and declare $v^{k*} \approx v^{k,j+1}$. Otherwise set $j = j+1$ and go to Step 1.

This gradient projection algorithm in Hilbert space has known convergence properties. In fact the following result obtains:

Theorem 9 *If $DVIC(F, f, \Gamma, D, U, x^0)$ is regular in the sense of Definition 5 while the conditions*

$$\langle v - v' + \lambda^T \left[\nabla_v f(x, v, v_D, t) - \nabla_v f(x, v', v'_D, t)\right], v - v'\rangle \geq \xi \|v - v'\| \tag{34}$$

and

$$\left\|v - v' + \lambda^T \left[\nabla_v f(x, v, v_D, t) - \nabla_v f(x, v', v'_D, t)\right]\right\| \leq \delta \|v - v'\| \tag{35}$$

are satisfied for some $\xi, \delta \in \Re^1_{++}$ and all $v, v' \in U$, then the gradient projection algorithm for the fixed point sub-problem converges.

Proof. Note that

$$\nabla_v J^k(v) = v - u^k + \alpha F\left(x^k, u^k, u^k_D, t\right) + \lambda^T \nabla_v f(x, v, vD, t)$$

From (34) we have

$$\Big\langle v - u^k + \alpha F\left(x^k, u^k, u^k_D, t\right) + \lambda^T \nabla_v f(x, v, v_D, t) -$$

$$\left[v' - u^k + \alpha F\left(x^k, u^k, t\right) + \lambda^T \nabla_v f(x, v', v'_D, t)\right], v - v'\Big\rangle \geq \xi \|v - v'\|$$

or

$$\langle \nabla_v J^k(v) - \nabla_v J^k(v'), v - v'\rangle \geq \xi \|v - v'\|$$

which is recognized as a coerciveness condition. Also (35) can be similarly re-stated as

$$\left\|\nabla_v J^k(v) - \nabla_v J^k(v')\right\| \leq \delta \|v - v'\|$$

which is recognized as a condition. Of course

$$v^{k,j+1} - v^{k*} = P_U\left[v^{k,j} - \theta_k \nabla_v J^k\left(v^{k,j}\right)\right] - P_U\left[v^{k*} - \theta_k \nabla_v J^k\left(v^{k*}\right)\right]$$

Because of the contractive nature of the projection operator, we have immediately that

$$\begin{aligned}
\left\|v^{k,j+1} - v^{k*}\right\|^2 &\leq \left\|v^{k,j} - v^{k*} - \theta_k\left(\nabla_v J^k\left(v^{k,j}\right) - \nabla_v J^k\left(v^{k*}\right)\right)\right\|^2 \\
&= \left\|v^{k,j} - v^{k*}\right\|^2 + (\theta_k)^2 \left\|\nabla_v J^k\left(v^{k,j}\right) - \nabla_v J^k\left(v^{k*}\right)\right\|^2 \\
&\quad -2\theta_k \langle \nabla_v J^k\left(v^{k,j}\right) - \nabla_v J^k\left(v^{k*}\right), v^{k,j} - v^{k*}\rangle
\end{aligned}$$

Because of coerciveness and the Lipschitz assumption, we have

$$\begin{aligned}
\left\|v^{k,j+1} - v^{k*}\right\|^2 &\leq \left\|v^{k,j} - v^{k*}\right\|^2 + (\theta_k\delta)^2 \left\|v^{k,j} - v^{k*}\right\|^2 - 2\theta_k\xi \left\|v^{k,j} - v^{k*}\right\|^2 \\
&= \left[1 + (\theta_k\delta)^2 - 2\theta_k\xi\right] \left\|v^{k,j} - v^{k*}\right\|^2
\end{aligned}$$

We may select θ_k such that $1 + (\theta_k\delta)^2 - 2\theta_k\xi < 1$ which is equivalent to a non-zero step obeying $\theta_k < \frac{2\xi}{\delta^2}$,a condition ensuring the algorithm is a strict contraction mapping. ∎

3 Brief Overview of Friesz, Bernstein, Suo and Tobin (2001) DUE Model

Most of the dynamic network user equilibrium (DUE) models proposed to date are comprised of four essential submodels:

1. a model of path delay;
2. flow dynamics;
3. flow propagation constraints; and
4. a route/departure-time choice model.

Peeta and Ziliaskopoulos (2001), in a comprehensive review of DTA and DUE research, note that there are several published models comprised of the four submodels named above.

3.1 Choice of Formulation

Recently Friesz and Mookherjee (2006) have shown how the DUE formulations by Friesz et al. (1993) and Friesz et al. (2001) may be numerically solved using infinite dimensional mathematical programming and a fixed point algorithm in Hilbert space. The Friesz et al. (1993) and Friesz et al. (2001) formulations are more computationally demanding than most if not all other DUE models because of the complicated path delay operators, equations of motion and time lags they embody. As such the algorithmic results they report and which are reviewed in this paper should work as well or better when adapted to other DUE models, including those for which path delay is determined by a nonlinear response surface or by simulation for a so-called rolling horizon. In the balance of this subsection, we closely follow Friesz et al. (2001) in presenting the DUE formulation emphasized in this paper.

The network of interest will form a directed graph $G(\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ denotes the set of nodes and $\mathcal{A}$ denotes the set of arcs; the respective cardianlities of these sets are $|\mathcal{N}|$ and $|\mathcal{A}|$. An arbitrary path $p \in \mathcal{P}$ of the network is

$$p \equiv \left\{a_1, a_2, ..., a_i, ..., a_{m(p)}\right\}$$

where $\mathcal{P}$ is the set of all paths and $m(p)$ is the number of arcs of p. We also let t_e denote the time at which flow exists an arc, while t_d is the time of departure from the origin of the same flow. The exit time function $\tau^p_{a_i}$ therefore obeys

$$t_e = \tau^p_{a_i}(t_d)$$

The relevant arc dynamics are

$$\begin{aligned} \frac{dx^p_{a_i}(t)}{dt} &= g^p_{a_{i-1}}(t) - g^p_{a_i}(t) \qquad \forall p \in \mathcal{P}, \quad i \in \{1, 2, ..., m(p)\} \\ x^p_{a_i}(t) &= x^p_{a_{i,0}} \qquad \forall p \in \mathcal{P}, \quad i \in \{1, 2, ..., m(p)\} \end{aligned}$$

where $x^p_{a_i}$ is the traffic volume of arc a_i contributed by path p, $g^p_{a_i}$ is flow exiting arc a_i and $g^p_{a_{i-1}}$ is flow entering arc a_i of path $p \in \mathcal{P}$. Also, $g^p_{a_0}$ is the flow exiting the origin of path p; by convention we call this the flow of path p and use the symbolic name

$$h_p = g^p_{a_0}$$

Furthermore

$$\delta_{a_i p} = \begin{cases} 1 & \text{if } a_i \in p \\ 0 & \text{if } a_i \notin p \end{cases}$$

so that

$$x_a(t) = \sum_{p \in \mathcal{P}} \delta_{ap} x^p_a(t) \qquad \forall a \in \mathcal{A}$$

is the total arc volume.

Arc unit delay is $D_a(x_a)$ for each arc $a \in \mathcal{A}$. That is, arc delay depends on the number of vehicles in front of an auto as that auto enters an arc. Of course total path traversal time is

$$D_p(t) = \sum_{i=1}^{m(p)} \left[\tau^p_{a_i}(t) - \tau^p_{a_{i-1}}(t)\right] = \tau^p_{a_{m(p)}}(t) - t \qquad \forall p \in \mathcal{P}$$

It is expedient to introduce the following recursive relationships that must hold in light of the above development:

$$\begin{aligned} \tau^p_{a_1}(t) &= t + D_{a_1}[x_{a_1}(t)] \quad \forall p \in \mathcal{P} \\ \tau^p_{a_i}(t) &= \tau^p_{a_{i-1}}(t) + D_{a_i}\left[x_{a_i}(\tau^p_{a_{i-1}}(t))\right] \quad \forall p \in \mathcal{P}, \quad i \in \{2, 3, ..., m(p)\} \end{aligned}$$

from which we have the nested path delay operators first proposed by Friesz et al. (1993):

$$D_p(t, x) \equiv \sum_{i=1}^{m(p)} \delta_{a_i p} \Phi_{a_i}(t, x) \quad \forall p \in \mathcal{P},$$

where

$$x = (x^p_{a_i} : p \in \mathcal{P}, i \in \{1, 2, ..., m(p)\}$$

and

$$\begin{aligned} \Phi_{a_1}(t, x) &= D_{a_1}(x_{a_1}(t)) \\ \Phi_{a_2}(t, x) &= D_{a_2}(x_{a_2}(t + \Phi_{a_1})) \\ \Phi_{a_3}(t, x) &= D_{a_3}(x_{a_3}(t + \Phi_{a_1} + \Phi_{a_2})) \\ &\vdots \\ \Phi_{a_i}(t, x) &= D_{a_i}(x_{a_i}(t + \Phi_{a_1} + \cdots + \Phi_{a_{i-1}})) \\ &= D_{a_i}(x_{a_i}(t + \sum_{j=1}^{i-1} \Phi_{a_j})). \end{aligned}$$

To ensure realistic behavior, we employ asymmetric early/late arrival penalties

$$F[t + D_p(t, x) - t_A]$$

where t_A is the desired arrival time and

$$\begin{aligned} t + D_p(t, x) > t_A &\Longrightarrow \quad F(t + D_p(t, x) - t_A) = \chi^L(x, t) > 0 \\ t + D_p(t, x) < t_A &\Longrightarrow \quad F(t + D_p(t, x) - t_A) = \chi^E(x, t) > 0 \\ t + D_p(t, x) = t_A &\Longrightarrow \quad F(t + D_p(t, x) - t_A) = 0 \\ \chi^L(t, x) > \chi^E(t, x) & \end{aligned}$$

We now combine the actual path delays and arrival penalties to obtain the *effective delay operators*

$$\Psi_p(t, x) = D_p(t, x) + F\{t + D_p(x, t) - T_A\} \qquad \forall p \in \mathcal{P} \tag{36}$$

Since the volume which enters and exits an arc should satisfy the conservation law, we must have

$$\int_0^t g^p_{a_{i-1}}(t)\, dt = \int_{D_{a_i}(x_{a_i}(0))}^{t + D_{a_i}(x_{a_i}(t))} g^p_{a_i}(t)\, dt \quad \forall p \in \mathcal{P}, i \in [1, m(p)] \tag{37}$$

where $g^p_{a_0}(t) = h_p(t)$. Differentiating the both sides of (37) with respect to time t and using the chain rule, we have

$$\begin{aligned} h_p(t) &= g^p_{a_1}(t + D_{a_1}(x_{a_1}(t)))(1 + D'_{a_1}(x_{a_1}(t))\dot{x}_{a_1}) \qquad \forall p \in \mathcal{P} \\ g^p_{a_{i-1}}(t) &= g^p_{a_i}(t + D_{a_i}(x_{a_i}(t)))(1 + D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i}) \qquad \forall p \in \mathcal{P}, \quad i \in [2, m(p)] \end{aligned}$$

These are *proper flow progression constraints* derived in a fashion that make them completely *consistent with the chosen dynamics and point queue model of arc delay.* These constraints involve a state dependent time lag $D_{a_i}(x_{a_i}(t))$ but make no explicit reference to the exit time functions. These flow propagation constraints describe the expansion and contraction of vehicle platoons; they were first presented by Friesz, Tobin, Bernstein and Suo (1995), Astarita (1995), Astarita (1996) independently proposed flow propagation constraints that may be readily placed in the above form.

3.2 Recast of DUE as a DVI with State Dependent Time Shifts

Given the traveling cost Θ_p for path p, the infinite dimensional variational inequality formulation for dynamic network user equilibrium itself is: find $(g^*, h^*) \in \Omega$ such that

$$\langle \Theta(t, x(h^*)), (h - h^*) \rangle = \sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Theta_p[t, x(h^*)] \left[h_p(t) - h_p^*(t) \right] dt \geq 0 \tag{38}$$

for all $(g, h) \in \Omega$, all of whose solutions **?** show are dynamic user equilibria[1]. In particular the solutions of (38) obey

$$\Theta_p(t, x^*) > \mu_{ij} \Longrightarrow h_p^*(t) = 0 \tag{39}$$

$$h_p^*(t) > 0 \Longrightarrow \Theta_p(t, x^*) = \mu_{ij} \tag{40}$$

for $p \in \mathcal{P}_{ij}$ where μ_{ij} is the lower bound on achievable costs for any ij-traveler, given by

$$\mu_p = \operatorname{ess\,inf} \{ \Theta_p(t, x) : t \in [t_0, t_f] \} \geq 0$$

and

$$\mu_{ij} = \min \left\{ \mu_p : p \in \mathcal{P}_{ij} \right\} \geq 0$$

We call a flow pattern satisfying (39) and (40) a *dynamic user equilibrium.* The behavior described by (39) and (40) is readily recognized to be a type of Cournot-Nash non-cooperative equilibrium. It is important to note that these conditions do not describe a stationary state, but rather a time varying flow pattern that is a Cournot-Nash equilibrium (or user equilibrium) at each instant of time.

4 Extensions

4.1 Dual Time Scales (day-to-day and within-day)

Let $\tau \in \Upsilon \equiv \{1, 2, ..., L\}$ be one typical day within the planning horizon, and take the length of each day to be Δ, while the clock time within each day τ is presented by $t \in [(\tau - 1)\Delta, \tau\Delta]$ for all $\tau \in \{1, 2, ..., L\}$. The planning horizon consists of L consecutive days. We assume the travel demand for each day changes based on the moving average of congestion experienced over previous days. We postulate that the travelling demand Q_{ij}^τ for day τ between a given O-D pair $(i, j) \in \mathcal{W}$ determined by the following system of difference equations:

$$Q_{ij}^{\tau+1} = \left[Q_{ij}^\tau - \eta_{ij}^\tau \left\{ \frac{\displaystyle\sum_{p \in \mathcal{P}_{ij}} \sum_{j=0}^{\tau-1} \int_{j \cdot \Delta}^{(j+1)\cdot\Delta} \Psi_p[t, x(h^*, g^*)] \, dt}{|\mathcal{P}_{ij}| \cdot \tau \cdot \Delta} - \chi_{ij} \right\} \right]^+ \quad \forall \tau \in \{1, 2, ...L - 1\} \tag{41}$$

$$Q_{ij}^1 = \tilde{Q}_{ij}$$

where $\tilde{Q}_{ij} \in \mathbb{R}_+$ is the fixed traveling demand for the O-D pair $(i, j) \in \mathcal{W}$ for the first day. The operator $[x]^+$ is equivalent to $\max[0, x]$.

4.2 Uncertain Travel Demand Information

Once again let us assume $\tau \in \Upsilon \equiv \{1, 2, ..., L\}$ be one typical day within the planning horizon, and take the length of each day to be Δ, while the clock time within each day τ is presented by $t \in [(\tau - 1)\Delta, \tau\Delta]$ for all $\tau \in \{1, 2, ..., L\}$. where the planning horizon consists of L consecutive days. Here we assume that the travel demand for each day is a random variable in the following multiplicative form

$$\hat{Q}_{ij}^\tau = Q_{ij}^\tau \cdot z_{ij}$$

[1]Although we have purposely supressed the functional analysis subtleties of the formulation, it should be noted that (38) involves an inner product in a Hilbert space, namely $\left(L^2[0, T]\right)^{|\mathcal{P}|}$.

where $\hat{Q}_{ij}^{\tau}$ is the realized travel demand on day τ between the OD pair (i,j) where as z_{ij} is the random variable. To keep exposition simple we assume that distribution of z_{ij} is known exactly, however, it can further be generalized to have only partial information (e.g., first and second moments) about z_{ij}. The average travel volume, Q_{ij}^{τ} may be computed from (41).

5 Numerical Example

In what follows, we consider a 5 arc, 4 node traffic network shown below. The forward star array and arc delay functions $D_a(x_a(t))$ for all 5 arcs of the network are contained in the following table:

Arc name	From node	To node	Arc Delay, $D_a(x_a(t))$
a_1	1	2	$\frac{1}{2}+\frac{x_{a_1}}{70}$
a_2	1	3	$1+\frac{x_{a_2}}{150}$
a_3	2	3	$\frac{1}{2}+\frac{x_{a_3}}{100}$
a_4	2	4	$1+\frac{x_{a_4}}{150}$
a_5	3	4	$\frac{1}{2}+\frac{x_{a_5}}{100}$

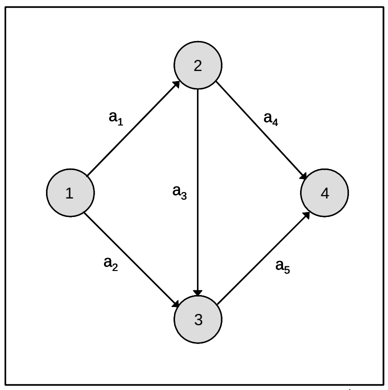

Fig 1 : The 5-arc 4-node traffic network with $(1,4)$ being the OD-pair

There is a travel demand of $Q_{14}^{1}=75$ units from node 1 (origin) to node 4 (destination) on day 1. There are 3 paths connecting nodes 1 through 4, namely

$$\begin{aligned} \mathcal{P}_{14} &= \{p_1,p_2,p_3\} \\ p_1 &= \{a_1,a_4\} \\ p_2 &= \{a_2,a_5\} \\ p_3 &= \{a_1,a_3,a_5\} \end{aligned}$$

We consider the planning horizon to be 4 days (i.e., $L=4$) and the length of each day is $\Delta=24$ hours. The desired arrival time for commuters is $T_A=13$ (1:00 PM of every day).The controls (path flows and arc exit flows) and states (arc traffic volumes) are enumerated in the following table:

Paths	Path Flows	Arc Exit Flows	Traffic Volume of Arcs
p_1	h_{p_1}	$g_{a_1}^{p_1}, g_{a_4}^{p_1}$	$x_{a_1}^{p_1}, x_{a_4}^{p_1}$
p_2	h_{p_2}	$g_{a_2}^{p_2}, g_{a_5}^{p_2}$	$x_{a_2}^{p_2}, x_{a_5}^{p_2}$
p_3	h_{p_2}	$g_{a_1}^{p_3}, g_{a_3}^{p_3}, g_{a_5}^{p_3}$	$x_{a_1}^{p_3}, x_{a_3}^{p_3}, x_{a_5}^{p_3}$

We consider the symmetric early/late arrival penalty

$$F\left[t+D_p(x,t)-T_A\right]=\left[t+D_p(x,t)-T_A\right]^2$$

Furthermore, without any loss of generality, we take the initial traffic volumes on every arc to be zero:

$$x_{a_i}^{p}(0)=0 \qquad \forall p\in\mathcal{P}, i\in[1,m(p)]$$

We forgo the detailed symbolic statement of this example and instead provide numerical results in graphical form for an essentially exact solution achieved after 29 iterations of the fixed point algorithm. Figures 2, 3 and 4 depict departure rates and arc exit flows for paths p_1, p_2 and p_3 respectively.

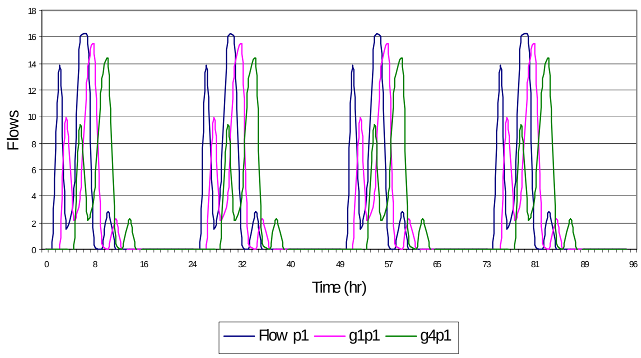

Fig 2 : Path and arc exit flows for path 1

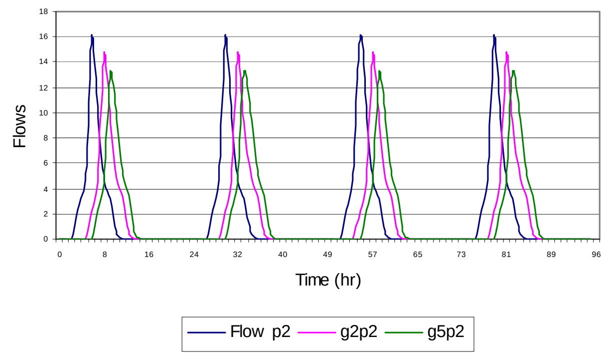

Fig 3 : Path and arc exit flows for path 2

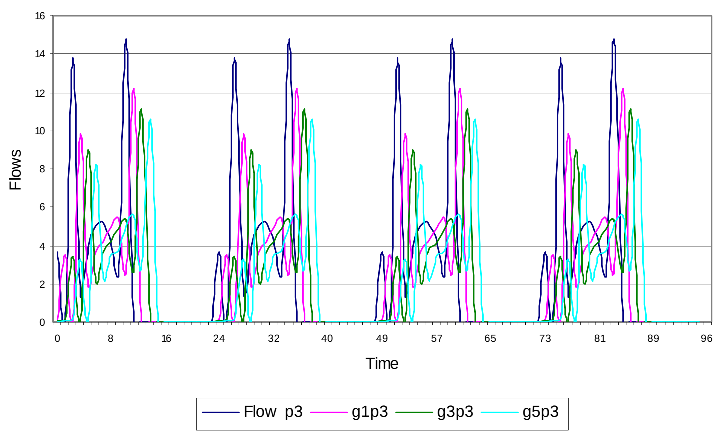

Fig 4 : Path and arc exit flows for path 3

Cumulative traffic volumes on the 5 different arcs are plotted against time in Figure 5 where

$$
\begin{aligned}
x_{a_1}(t) &= x_{a_1}^{p_1}(t) + x_{a_1}^{p_3}(t) \\
x_{a_2}(t) &= x_{a_2}^{p_2}(t) \\
x_{a_3}(t) &= x_{a_3}^{p_3}(t) \\
x_{a_4}(t) &= x_{a_4}^{p_1}(t) \\
x_{a_5}(t) &= x_{a_5}^{p_2}(t) + x_{a_5}^{p_3}(t)
\end{aligned}
$$

for all time $t \in [0, L\Delta]$.

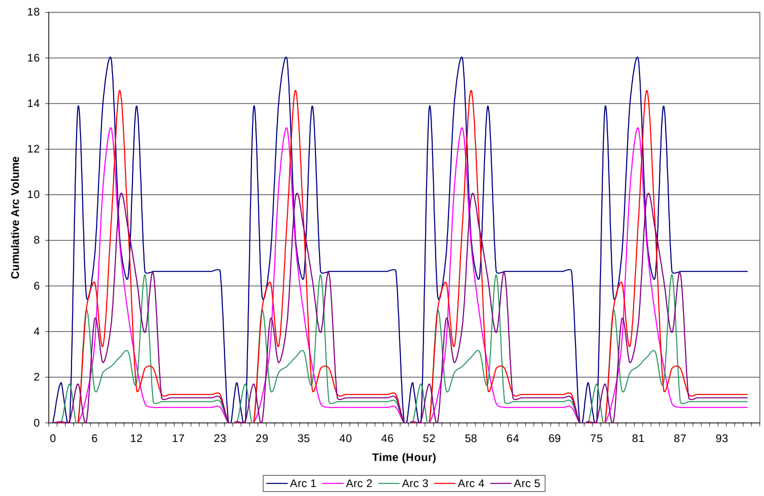

Fig 5 : Cumulative arc volume vs. time

Note that the effective path delay operator in (36) gives the unit travel cost along a path p at time t. Figure 6 analyzes the effective delay and flow for path p_2 by plotting both for the same time scale which shows that path flow is maximal when the associated unit travel cost (effective path delay) is at its well defined minimum.

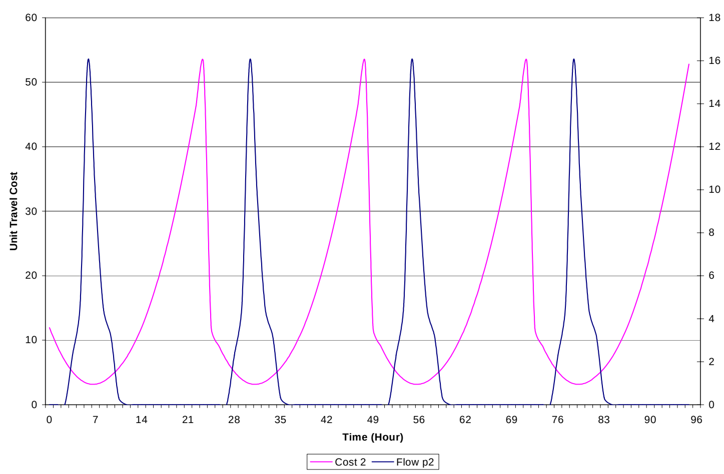

Fig 6 : Comparison of path flows and associated unit travel costs for path p_2

Net travel demand and demand reduction are plotted below against the same time scale (day) which clearly demonstrates that more commuters switch to alternative mode (e.g., telecommuting) as their rolling average experience of congestion increases with passage of time.

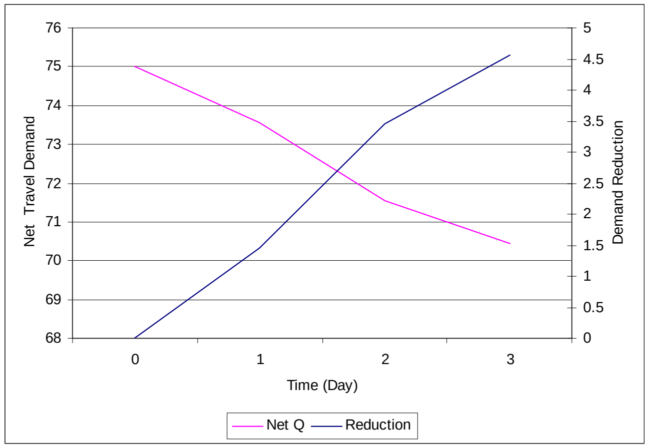

Fig 7 : Net travel demand and demand reduction

6 Concluding Remarks

We have explained how traditional non-cooperative differential game theory may be extended to accomodate the natural formulation of DUE as an infinite dimensional variational inequality involving explicit state-dependent time shifts. We show that such a perspective is not only useful for analysis but also leads to simple yet effective algorithms for the computation of DUE solutions. We also apply this formalism to create two entirely new formulations of dynamic user equilibrium when: (1) there are dual time scales (day-to-day and within-day); and (2) demand information is uncertain. Our future DUE reserach will provide a more in-depth analysis of the stochastic DUE problem in the presence of incomplete traffic information.

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