A Computable Theory of Dynamic Congestion Pricing*

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Abstract

In this paper we present a theory of dynamic congestion pricing for the day-to-day as well as the within-day time scales. The equilibrium design problem emphasized herein takes the form of an MPEC, which we call the Dynamic Optimal Toll Problem with Equilibrium Constraints, or DOTPEC. The DOPTEC formulation we employ recalls an important earlier result that allows the equilibrium design problem to be stated as a single level problem, a result which is surprisingly little known. The DOPTEC maintains the usual design objective of minimizing the system travel cost by appropriate toll pricing. We describe how an infnite dimensional mathematical programming perspective may be employed to create an algorithm for the DOTPEC. A numerical example is provided.

Keywords: Dynamic congestion pricing; Dynamic user equilibrium; Differential Variational Inequality; Optimal Control

1 Introduction

The advent of new commitments by municipal, state and federal governments to construct and operate roadways whose tolls may be set dynamically has brought into sharp focus the need for a computable theory of dynamic tolls. Moreover, it is clear from the policy debates that surround the issue of dynamic tolls that pure economic efficiency is not the sole or even the most prominent objective of any dynamic toll mechanism that will be implemented. Rather, equity considerations as well as preferential treatment for certain categories of commuters must be addressed by such a mechanism. Accordingly, we introduce in this paper the dynamic user equilibrium optimal toll problem and discuss two plausible algorithms for its solution; we also provide detailed numerical results that document the performance of the two algorithms.

The dynamic user equilibrium optimal toll problem should not be considered a simple dynamic extension of the traditional congestion pricing paradigm associated with static user equilibrium and usually accredited to Beckmann et al. (1955). Rather, the dynamic user equilibrium optimal toll problem is most closely related to the equilibrium network design problem which is now widely recognized to be a specific instance of a mathematical program with equilibrium constraints (MPEC). In fact it will be convenient to refer to the dynamic user equilibrium optimal toll problem as the dynamic optimal toll problem with equilibrium constraints or DOTPEC, where it is understood that the equilibrium of interest is a dynamic user equilibrium.

The relevant background literature for the DOTPEC includes a paper by Friesz et al. (2002) who discuss a version of the DOTPEC but for the day-to-day time scale rather than the dual (within-day as well as day-to-day) time scale formulation emphasized in this paper. Also pertinent are the paper by Friesz et al. (1996) which discusses dynamic disequilibrium network design and the review

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by Liu (2004) which considers multi-period efficient tolls. Although the DOTPEC is not the same as the problem of determining efficient tolls including the latter's multiperiod generalization, the exact nature of the differences and similarities is not known and has never been studied. To study the DOTPEC, it is necessary to employ some form of dynamic user equilibrium model. We elect the formulation due to Friesz et al. (2001), Friesz and Mookherjee (2006) and its varieties analyzed by Ban et al. (2005) and others. The dynamic efficient toll formulation will be constructed by direct analogy to the static efficient toll problem formulation of Hearn and Yildirim (2002).

The main focus of this paper is the formulation and solution of the DOTPEC. To this end, again using the DUE formulation reported in Friesz et al. (2001) and Friesz and Mookherjee (2006), we will form a Stackelberg game that envisions a central authority minimizing social costs through its control of link tolls subject to DUE constraints with the potential for additional side constraints for equity and other policy considerations. Also, since we will allow multiple target arrival times of the users, the within-day scale model, we show how to easily extend the formulation to include the day-to-day evolution of demand. Of course there are several ways such a model may be formulated. The dual-time scale formulation we shall emphasize is based on our prior work on differential variational inequalities and equilibrium network design and follows the qualitative theory conjectured (but not analyzed) by Friesz et al. (1996).

Central to the study of the DOTPEC in this paper is the dynamic generalization of a result due to Tan et al. (1979) and reprised by Friesz and Shah (2001) showing that a system of inequalities expressing the relationship of average effective delay to minimum delay is equivalent to a static user equilibrium. This system of inequalities allows one to state the equilibrium network design problem as a single level mathematical program. Extension of this result to a dynamic setting allows us in this paper to state the DOTPEC as an equivalent, non-hierarchical optimal control problem. We consider two principal methods for solving this optimal control problem: (1) descent in Hilbert space without time discretization, and (2) a finite dimensional approximation solved as a nonlinear program. In both approaches we employ an implicit fixed point scheme like that in Friesz and Mookherjee (2006) for dealing with time shifts in differential variational inequalities. In an example provided near the end of this paper, we numerically study a small network and determine its optimal dynamic tolls.

2 Notation and Model Formulation

In this section we purposely repeat key portions of the time-lagged DUE formulation given in Friesz et al. (2001), because of its key role in this manuscript. The reader familiar with the notation and time-shifted DUE model presented in Friesz et al. (2001) may skip this section of the present paper.

2.1 Dynamics, Delay Operators and Constraints

The network of interest will form a directed graph $G(\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ denotes the set of nodes and $\mathcal{A}$ denotes the set of arcs; the respective cardinalities of these sets are $|\mathcal{N}|$ and $|\mathcal{A}|$. An arbitrary path $p \in \mathcal{P}$ of the network is

$$p \equiv \{a_1, a_2, ..., a_i, ..., a_{m(p)}\}$$

where $\mathcal{P}$ is the set of all paths and $m(p)$ is the number of arcs of p. We also let t_e denote the time at which flow exists an arc, while t_d is the time of departure from the origin of the same flow. The exit time function $\tau^p_{a_i}$ therefore obeys

$$t_e = \tau^p_{a_i}(t_d)$$

The relevant arc dynamics are

$$\frac{dx_{a_i}^p(t)}{dt} = g_{a_{i-1}}^p(t) - g_{a_i}^p(t) \qquad \forall p \in \mathcal{P}, \quad i \in \{1, 2, ..., m(p)\} \tag{1}$$

$$x_{a_i}^p(t) = x_{a_{i,0}}^p \qquad \forall p \in \mathcal{P}, \quad i \in \{1, 2, ..., m(p)\} \tag{2}$$

where $x_{a_i}^p$ is the traffic volume of arc a_i contributed by path p, $g_{a_i}^p$ is flow exiting arc a_i and $g_{a_{i-1}}^p$ is flow entering arc a_i of path $p \in \mathcal{P}$. Also, $g_{a_0}^p$ is the flow exiting the origin of path p; by convention we call this the flow of path p and use the symbolic name

$$h_p = g_{a_0}^p$$

Furthermore

$$\delta_{a_i p} = \begin{cases} 1 & \text{if } a_i \in p \\ 0 & \text{if } a_i \notin p \end{cases}$$

so that

$$x_a(t) = \sum_{p \in \mathcal{P}} \delta_{ap} x_a^p(t) \qquad \forall a \in \mathcal{A}$$

is the total arc volume.

Arc unit delay is $D_a(x_a)$ for each arc $a \in \mathcal{A}$. That is, arc delay depends on the number of vehicles in front of a vehicle as it enters an arc. Of course total path traversal time is

$$D_p(t) = \sum_{i=1}^{m(p)} \left[\tau_{a_i}^p(t) - \tau_{a_{i-1}}^p(t)\right] = \tau_{a_{m(p)}}^p(t) - t \qquad \forall p \in \mathcal{P}$$

It is expedient to introduce the following recursive relationships that must hold in light of the above development:

$$\begin{aligned} &\tau_{a_1}^p(t) = t + D_{a_1}\left[x_{a_1}(t)\right] \quad \forall p \in \mathcal{P} \\ &\tau_{a_i}^p(t) = \tau_{a_{i-1}}^p(t) + D_{a_i}\left[x_{a_i}(\tau_{a_{i-1}}^p(t))\right] \quad \forall p \in \mathcal{P}, \quad i \in \{2, 3, ..., m(p)\} \end{aligned}$$

from which we have the nested path delay operators first proposed by Friesz et al. (1993):

$$D_p(t, x) \equiv \sum_{i=1}^{m(p)} \delta_{a_i p} \Phi_{a_i}(t, x) \quad \forall p \in \mathcal{P},$$

where

$$x = (x_{a_i}^p : p \in \mathcal{P}, i \in \{1, 2, ..., m(p)\}$$

and

$$\begin{aligned} \Phi_{a_1}(t, x) &= D_{a_1}(x_{a_1}(t)) \\ \Phi_{a_2}(t, x) &= D_{a_2}(x_{a_2}(t + \Phi_{a_1})) \\ \Phi_{a_3}(t, x) &= D_{a_3}(x_{a_3}(t + \Phi_{a_1} + \Phi_{a_2})) \\ &\vdots \\ \Phi_{a_i}(t, x) &= D_{a_i}(x_{a_i}(t + \Phi_{a_1} + \cdots + \Phi_{a_{i-1}})) \\ &= D_{a_i}\left(x_{a_i}\left(t + \sum_{j=1}^{i-1} \Phi_{a_j}\right)\right). \end{aligned}$$

To ensure realistic behavior, we employ asymmetric early/late arrival penalties

$$F\left[t+D_p\left(t,x\right)-t_A\right]$$

where t_A is the desired arrival time and

$$\begin{array}{rl} t+D_p(t,x)>t_A \Longrightarrow & F(t+D_p(t,x)-t_A)=\chi^L(x,t)>0 \\ t+D_p(t,x)<t_A \Longrightarrow & F(t+D_p(t,x)-t_A)=\chi^E(x,t)>0 \\ t+D_p(t,x)=t_A \Longrightarrow & F(t+D_p(t,x)-t_A)=0 \end{array}$$

while

$$\chi^L(t,x)>\chi^E(t,x)$$

Let us further denote arc tolls by y_a for each arc $a\in\mathcal{A}$. We assume that users pay any toll imposed on an arc at the entrance of the arc. Then the path tolls y_p for each path $p\in\mathcal{P}$ are

$$y_p\left(t\right)=\sum_{i=1}^{m(p)}\delta_{a_ip}y_{a_i}\left(t+\sum_{j=1}^{i-1}\Phi_{a_j}(t,x)\right)\quad\forall p\in\mathcal{P}$$

where $\Phi_{a_0}(t,x)=0$. If the tolls are paid when users exit arcs, then the path toll becomes

$$y_p\left(t\right)=\sum_{i=1}^{m(p)}\delta_{a_ip}y_{a_i}\left(t+\sum_{j=1}^{i}\Phi_{a_j}(t,x)\right)\quad\forall p\in\mathcal{P}$$

We now combine the actual path delays and arrival penalties to obtain the *effective delay operators*

$$\Psi_p(t,x)=D_p(t,x)+F\left(t+D_p(x,t)-T_A\right)\qquad\forall p\in\mathcal{P}\tag{3}$$

Since the volume which enters and exits an arc should conserve flow, we must have

$$\int_0^t g^p_{a_{i-1}}\left(t\right)dt=\int_{D_{a_i}(x_{a_i}(0))}^{t+D_{a_i}(x_{a_i}(t))}g^p_{a_i}\left(t\right)dt\quad\forall p\in\mathcal{P},i\in[1,m(p)]\tag{4}$$

where $g^p_{a_0}(t)=h_p(t)$. Differentiating both sides of (4) with respect to time t and using the chain rule, we have

$$h_p\left(t\right)=g^p_{a_1}(t+D_{a_1}(x_{a_1}(t)))(1+D'_{a_1}(x_{a_1}(t))\dot{x}_{a_1})\qquad\forall p\in\mathcal{P}\tag{5}$$

$$g^p_{a_{i-1}}\left(t\right)=g^p_{a_i}(t+D_{a_i}(x_{a_i}(t)))(1+D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i})\qquad\forall p\in\mathcal{P},\quad i\in[2,m(p)]\tag{6}$$

These are *proper flow progression constraints* derived in a fashion that makes them completely *consistent with the chosen dynamics and point queue model of arc delay.* These constraints involve a state-dependent time lag $D_{a_i}(x_{a_i}(t))$ but make no explicit reference to the exit time functions. These flow propagation constraints describe the expansion and contraction of vehicle platoons; they were presented by Friesz et al. (1995). Astarita (1995,1996) independently proposed flow propagation constraints that may be readily placed in the above form.

The final constraints to consider are those of flow conservation and non-negativity:

$$\sum_{p\in\mathcal{P}_{ij}}\int_{t_0}^{t_f}h_p(t)dt=Q_{ij}\quad\forall\left(i,j\right)\in\mathcal{W}\tag{7}$$

$$h_p\geq 0\quad\forall\left(i,j\right)\in\mathcal{P}_{ij}\tag{8}$$

$$g_{a_i}^p \geq 0 \quad \forall p \in \mathcal{P}, i \in [1, m(p)] \tag{9}$$

$$g_{a_i}^p \geq 0 \quad \forall p \in \mathcal{P}, i \in [1, m(p)] \tag{10}$$

where $\mathcal{W}$ is the set of origin-destination pairs, $\mathcal{P}_{ij}$ is the set of paths connecting origin-destination pair (i, j), $t_f > t_0$, and $t_f - t_0$ defines the planning horizon. Furthermore, Q_{ij} is the travel demand (a volume) for the period $[t_0.t_f]$. In what follows h will denote the vector of all path flows, g the vector of all arc exit flows. Finally, we denote the set of all feasible exit flow vectors (h, g) by Ω; that is

$$\Omega \equiv \{(h, g) : (1), (2), (5), (6), (7), (8), (9), (10) \text{ are satisfied}\} \tag{11}$$

2.2 Dynamic User Equilibrium

Given the effective unit travel delay Ψ_p for path p, the infinite dimensional variational inequality formulation for dynamic network user equilibrium itself is: find $(g^*, h^*) \in \Omega$ such that

$$\langle \Psi(t, x(h^*, g^*)), (h - h^*) \rangle = \sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p [t, x(h^*, g^*)] \cdot \left[h_p(t) - h_p^*(t)\right] dt \geq 0 \tag{12}$$

for all $(h, g) \in \Omega$, where Ψ denotes the vector of effective path delay operators. Friesz et al. (2001) show all solutions of (12) are dynamic user equilibria[1]. In particular the solutions of (12) obey

$$\Psi_p(t, x(g^*, h^*)) > \mu_{ij} \Longrightarrow h_p^*(t) = 0 \tag{13}$$

$$h_p^*(t) > 0 \Longrightarrow \Psi_p(t, x(g^*, h^*)) = \mu_{ij} \tag{14}$$

for $p \in \mathcal{P}_{ij}$ where μ_{ij} is the lower bound on achievable costs for any ij-traveler, given by

$$\mu_p = \operatorname{ess\,inf} \{\Theta_p(t, x) : t \in [t_0, t_f]\} \geq 0$$

and

$$\mu_{ij} = \min \{\mu_p : p \in \mathcal{P}_{ij}\} \geq 0$$

We call a flow pattern satisfying (13) and (14) a *dynamic user equilibrium.* The behavior described by (13) and (14) is readily recognized to be a type of Cournot-Nash non-cooperative equilibrium. It is important to note that these conditions do not describe a stationary state, but rather a time varying flow pattern that is a Cournot-Nash equilibrium (or user equilibrium) at each instant of time.

3 The Dynamic Efficient Toll Problem (DETP)

Hearn and Yildirim (2002) studied the efficient toll in the static setting with the traveling cost which is linear in the traffic flow. The objective of the efficient toll is to make the user equilibrium traffic flow equivalent to the system optimum by appropriate congestion pricing. To study the dynamic efficient toll problem (DETP), we introduce the notion of a *tolled effective delay operator*:

$$\Theta_p(t, x, y_p) = D_p(t, x) + F\{t + D_p(x, t) - T_A\} + y_p(t) \qquad \forall p \in \mathcal{P}$$

where y_p denotes the toll for path p. Of course we have the relationship

$$\Theta_p(t, x, y_p) = \Psi_p(t, x) + y_p(t) \tag{15}$$

[1] Although we have purposely supressed the functional analysis subtleties of the formulation, it should be noted that (12) involves an inner product in a Hilbert space, namely $\left(L^2[0, T]\right)^{|\mathcal{P}|}$.

3.1 Analysis of the System Optimum

The dynamic system optimum (DSO) is achieved by solving

$$\min \ J_1 = \int_{t_0}^{t_f} \sum_{p\in\mathcal{P}} e^{-rt}\Psi_p(t,x)\,h_p(t)\,dt$$

subject to

$$\frac{dx^p_{a_i}(t)}{dt} = g^p_{a_{i-1}}(t) - g^p_{a_i}(t) \qquad \forall p\in\mathcal{P}, \quad i\in\{1,2,...,m(p)\} \tag{16}$$

$$x^p_{a_i}(t) = x^p_{a_{i,0}} \qquad \forall p\in\mathcal{P}, \quad i\in\{1,2,...,m(p)\}$$

$$g^p_{a_{i-1}}(t) = g^p_{a_i}(t + D_{a_i}(x_{a_i}(t)))(1 + D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i}) \qquad \forall p\in\mathcal{P}, \quad i\in[1,m(p)] \tag{17}$$

$$\sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t)\,dt = Q_{ij} \quad \forall (i,j)\in\mathcal{W} \tag{18}$$

$$x\geq 0 \qquad g\geq 0 \qquad h\geq 0 \tag{19}$$

where we have used the convention

$$g^p_{a_0} = h_p$$

It will be convenient to employ the following shorthand for shifted variables:

$$\bar{g}^p_{a_i} \doteq g^p_{a_i}(t + D_{a_i}(x_{a_i}(t))) \quad \forall p\in\mathcal{P}, \quad i\in[0,m(p)]$$

Penaltizing (17) we obtain

$$J_1 = \int_{t_0}^{t_f} \left\{ \sum_{p\in\mathcal{P}} e^{-rt}\Psi_p(t,x)\,h_p(t) + \sum_{p\in\mathcal{P}} \sum_{i=1}^{m(p)} \frac{\mu^p_{a_i}}{2} \left[g^p_{a_{i-1}}(t) - \bar{g}^p_{a_i}(t)\left(1 + D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i}\right) \right]^2 \right\} dt \tag{20}$$

where $\mu^p_{a_i}$ is the penalty coefficient. Let us then define the set of feasible controls

$$\Lambda \equiv \left\{ (h,g) : \sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t)\,dt = Q_{ij} \ \ \forall (i,j)\in\mathcal{W}, h\geq 0, \quad g\geq 0 \right\} \tag{21}$$

Optimal control problem (20) and (21) is an instance of the time-shifted optimal control problem analyzed in Friesz et al. (2001). We also employ the following notation for the state vector and control vector, respectively:

$$x = \left(x^p_{a_i}\right)_{p\in\mathcal{P}, i\in[1,m(p)]}$$

$$g = \left(g^p_{a_i}\right)_{p\in\mathcal{P}, i\in[0,m(p)]}$$

The DSO Hamiltonian is

$$H_1(t,x,h,g,\lambda;\mu) \equiv \sum_{p\in\mathcal{P}} e^{-rt}\Psi_p(t,x)\,h_p(t) + \sum_{p\in\mathcal{P}} \sum_{i=1}^{m(p)} \frac{\mu^p_{a_i}}{2} \left\{ g^p_{a_{i-1}}(t) - \bar{g}^p_{a_i}(t)(1 + D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i}) \right\}^2$$
$$+ \sum_{p\in\mathcal{P}} \sum_{i=1}^{m(p)} \lambda^p_{a_i} \left(g^p_{a_{i-1}}(t) - g^p_{a_i}(t) \right)$$

Let us introduce the vector

$$F(t,x,h,g,\lambda;\mu) = \left(F^p_{a_i}(t,x,h,g,\lambda;\mu)\right)_{p\in\mathcal{P},i\in[0,m(p)]}$$

where

$$F^p_{a_0}(t,x,h,g,\lambda;\mu) = \frac{\partial H_1(t,x,h,g,\lambda;\mu)}{\partial h_p} \quad \forall p\in\mathcal{P} \tag{22}$$

$$F^p_{a_i}(t,x,h,g,\lambda;\mu) = \quad \forall p\in\mathcal{P}, \quad i\in[1,m(p)] \tag{23}$$

$$\frac{\partial H_1(t,x,h,g,\lambda;\mu)}{\partial g^p_{a_i}} \quad \text{if } t\in[t_0, D_{a_i}(x(t_0))]$$

$$\frac{\partial H_1(t,x,h,g,\lambda;\mu)}{\partial g^p_{a_i}} + \left[\frac{\partial H_1(t,x,h,g,\lambda;\mu)}{\partial \bar{g}^p_{a_i}}\frac{1}{1+D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i}}\right]_{s_{a_i}(t)} \quad \text{if } t\in[D_{a_i}(x(t_0)), t_f + D_{a_i}(x(t_f))]$$

and each $s_{a_i}(t)$ is a solution of the fixed point problem

$$s_{a_i}(t) = \arg\left[s = t - D_{a_i}(x(s))\right]$$

We may write (22) and (23) in detail as

$$\begin{aligned} F^p_{a_0}(t,x,h,g,\lambda;\mu) = e^{-rt}&\left[\Psi_p(t,x) + \frac{\partial \Psi_p(t,x)}{\partial h_p}h_p\right] \\ &+ \mu^p_{a_1}\left[g^p_{a_0}(t) - \bar{g}^p_{a_1}(t)(1+D'_{a_1}(x_{a_1}(t))\dot{x}_{a_1})\right] + \lambda^p_{a_1} \quad \forall p\in\mathcal{P} \end{aligned} \tag{24}$$

$$F^p_{a_i}(t,x,h,g,\lambda;\mu) = \quad \forall p\in\mathcal{P}, \quad i\in[1,m(p)-1]$$

$$\mu^p_{a_{i+1}}\left\{g^p_{a_i}(t) - \bar{g}^p_{a_{i+1}}(t)(1+D'_{a_{i+1}}(x_{a_{i+1}}(t))\dot{x}_{a_{i+1}})\right\} - \lambda^p_{a_i} + \lambda^p_{a_{i+1}} \quad \text{if } t\in[t_0, D_{a_i}(x(t_0))]$$

$$\begin{aligned} &\mu^p_{a_{i+1}}\left\{g^p_{a_i}(t) - \bar{g}^p_{a_{i+1}}(t)(1+D'_{a_{i+1}}(x_{a_{i+1}}(t))\dot{x}_{a_{i+1}})\right\} - \lambda^p_{a_i} + \lambda^p_{a_{i+1}} \\ &-\left[\mu^p_{a_i}\left\{g^p_{a_{i-1}}(t) - \bar{g}^p_{a_i}(t)(1+D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i})\right\}\right]_{s_{a_i}(t)} \quad \text{if } t\in[D_{a_i}(x(t_0)), t_f + D_{a_i}(x(t_f))] \end{aligned}$$

$$F^p_{a_i}(t,x,h,g,\lambda;\mu) = \quad \forall p\in\mathcal{P}, \quad i = m(p)$$

$$-\lambda^p_{a_i} \text{if } t\in[t_0, D_{a_i}(x(t_0))]$$

$$-\lambda^p_{a_i} - \left[\mu^p_{a_i}\left\{g^p_{a_{i-1}}(t) - \bar{g}^p_{a_i}(t)(1+D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i})\right\}\right]_{s_{a_i}(t)} \text{if } t\in[D_{a_i}(x(t_0)), t_f + D_{a_i}(x(t_f))]$$

Then a necessary condition for $(h^S, g^S)\in\Lambda$ to be the system optimum is

$$0 \le \sum_{p\in\mathcal{P}}\sum_{i=0}^{m(p)} F^p_{a_i}\left(t, x^S, h^S, g^S, \lambda^S;\mu\right)\left(g^p_{a_i} - g^{pS}_{a_i}\right) \qquad \forall (h,g)\in\Lambda \tag{25}$$

for each time instant $t\in\left[t_0, \sup_{a_i\in A}\{t_f + D_{a_i}(x(t_f))\}\right]$, together with the state dynamics (16) and the following adjoint equations and boundary conditions

$$-\frac{d\lambda^{p,S}_{a_i}}{dt} = \frac{\partial H^S_1}{\partial x^p_{a_i}} = e^{-rt}\frac{\partial \Psi_p\left(t,x^S\right)}{\partial x^p_{a_i}} \qquad \forall p\in\mathcal{P}, \quad i\in[1,m(p)]$$

$$\lambda^{p,S}_{a_i}(t_f) = 0 \qquad \forall p\in\mathcal{P}, \quad i\in[1,m(p)]$$

where the superscript S denotes a trajectory corresponding to a system optimum.

3.2 Analysis of the User Equilibrium in the Presence of Tolls

However, a dynamic tolled user equilibrium must obey

$$\sum_{p\in\mathcal{P}}\int_{t_0}^{t_f} e^{-rt}\left\{\Theta_p\left[t,x\left(h^U\right),y_p^U\right]\right\}\left[h_p(t)-h_p^U(t)\right]dt\geq 0 \quad \text{for all } (h,g)\in\Lambda \tag{26}$$

where the state dynamics as well as all other state and control constraints are identical to those introduced above for DSO. In particular, the set of feasible controls Λ referred to in (26) remains unchanged. We formulate an optimal control problem[2] from the above dynamic user equilibrium variational inequality problem; its objective is

$$\min\ J_2=\sum_{p\in\mathcal{P}}\int_{t_0}^{t_f} e^{-rt}\Theta_p\left[t,x\left(h^U\right),y_p^U\right]h_p(t)\,dt$$

with the same constraints introduced previously. As previously done for the system optimum problem, we penaltize the flow propogation constraints to obtain the modified criterion

$$J_2=\sum_{p\in\mathcal{P}}\int_{t_0}^{t_f}\left\{e^{-rt}\Theta_p\left[t,x\left(h^U\right),y_p^U\right]h_p(t)+\sum_{p\in\mathcal{P}}\sum_{i=1}^{m(p)}\frac{\mu_{a_i}^p}{2}\left[g_{a_{i-1}}^p(t)-\bar{g}_{a_i}^p(t)(1+D_{a_i}'(x_{a_i}(t))\dot{x}_{a_i})\right]^2\right\}dt \tag{27}$$

Then we have another standard form time-shifted optimal control problem, although it is subtely but importantly different than that for DSO. In particular, the Hamiltonian now becomes

$$\begin{aligned}H_2(t,x,h,g,\lambda;\mu)\equiv&\sum_{p\in\mathcal{P}}e^{-rt}\Theta_p\left[t,x\left(h^U\right),y_p^U\right]h_p(t)+\sum_{p\in\mathcal{P}}\sum_{i=1}^{m(p)}\frac{\mu_{a_i}^p}{2}\left\{g_{a_{i-1}}^p(t)-\bar{g}_{a_i}^p(t)(1+D_{a_i}'(x_{a_i}(t))\dot{x}_{a_i})\right\}^2\\&+\sum_{p\in\mathcal{P}}\sum_{i=1}^{m(p)}\lambda_{a_i}^p\left(g_{a_{i-1}}^p(t)-g_{a_i}^p(t)\right)\end{aligned}$$

An analysis of necessary conditions similar to that for DSO is now possible. The key difference is that the counterpart of (24) must in the user equilibrium case be written as follows:

$$\begin{aligned}G_{a_0}^p(t,x,h,g,\lambda;\mu)=&\ e^{-rt}\Theta_p\left[t,x\left(h^U\right),y_p^U\right]\\&+\mu_{a_1}^p\left[g_{a_0}^p(t)-\bar{g}_{a_1}^p(t)(1+D_{a_1}'(x_{a_1}(t))\dot{x}_{a_1})\right]+\lambda_{a_1}^p\quad\forall p\in\mathcal{P}\\G_{a_i}^p(t,x,h,g,\lambda;\mu)=&\ F_{a_i}^p(t,x,h,g,\lambda;\mu)\qquad\forall p\in\mathcal{P},\quad i\in[1,m(p)]\end{aligned}$$

Then a necessary condition for $\left(h^S,g^S\right)\in\Lambda$ to be a dynamic user equilibrium (DUE) is

$$0\leq\sum_{p\in\mathcal{P}}\sum_{i=0}^{m(p)}G_{a_i}^p\left(t,x^U,h^U,g^U,\lambda^U;\mu\right)\left(g_{a_i}^p-g_{a_i}^{pU}\right)\qquad g\in\Lambda \tag{28}$$

for each time instant $t\in\left[t_0,\sup_{a_i\in A}\left\{t_f+D_{a_i}\left(x\left(t_f\right)\right)\right\}\right]$, together with the state dynamics (16) and the following adjoint equations and boundary conditions:

$$-\frac{d\lambda_{a_i}^{p,U}}{dt}=\frac{\partial H_2^U}{\partial x_{a_i}^p}=e^{-rt}\frac{\partial\Theta_p\left[t,x\left(h^U\right),y_p^U\right]}{\partial x_{a_i}^p}\qquad\forall p\in\mathcal{P},\quad i\in[1,m(p)]$$

[2] may not be used for numerical computation as its statement depends on knowledge of the dynamic user equilibrium being sought. However, it may be employed for qualitative analyses like those which follow.

$$\lambda_{a_i}^{p,U}(t_f) = 0 \qquad \forall p \in \mathcal{P}, \quad i \in [1, m(p)]$$

where the superscript U denotes a trajectory corresponding to a dynamic user equilibrium in the presence of tolls.

3.3 Characterizing Efficient Tolls

It is the purpose of efficient tolls to make the criteria J_1 amd J_2 identical along solution trajectories for which flow propagation and other constraints are satisfied, for then the system optimal total costs are identical to the tolled user optimal total costs. Furthermore, the vectors of path flows (departure rates) obey

$$h^U(t) = h^S(t) \tag{29}$$

There are as well identical arc exit flows and identical arc volumes. Therefore, along solution trajectories

$$\lambda_{a_1}^{p,S} = \frac{\partial J_1}{\partial x_{a_1}^{p,S}} = \frac{\partial J_2}{\partial x_{a_1}^{p,U}} = \lambda_{a_1}^{p,U} \tag{30}$$

With (30) in mind and upon comparing (25) and (28), we find

$$\begin{aligned} e^{-rt}\left\{\Psi_p\left(t, x^S\right) + \frac{\partial \Psi_p\left(t, x^S\right)}{\partial h_p} h_p^S\right\} &= e^{-rt}\left\{\Theta_p\left[t, x\left(h^U\right), y_p^U\right]\right\} \\ &= e^{-rt}\left\{\Psi_p\left(t, x^U\right) + y_p^U(t)\right\} \end{aligned}$$

which may be immediately re-stated as the following decision rule:

$$y_p^U(t) = \frac{\partial \Psi_p\left(t, x^S\right)}{\partial h_p} h_p^S \qquad \forall t \in [t_0, t_f] \tag{31}$$

This result is completely analogous to that for an efficiently tolled static user equilibrium.

4 The Dynamic Optimal Toll Problem with Equilibrium Constraints (DOTPEC)

We now introduce the dynamic optimal toll problem with equilibrium constraints (DOTPEC). The DOTPEC is a type of dynamic network design problem for which a central authority seeks to minimize congestion in a transport network, whose flows obey a dynamic network user equilibrium, by dynamically adjusting tolls. In particular the central authority seeks to solve the optimal control problem

$$\min \; J = \int_{t_0}^{t_f} \sum_{p \in \mathcal{P}} \Psi_p(t, x)\, h_p(t)\, dt \tag{32}$$

subject to

$$\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Theta_p\left[t, x(h, g), y_p\right](w_p - h_p)\, dt \geq 0 \quad \forall (w, g) \in \Lambda \tag{33}$$

$$\frac{dx_{a_i}^p(t)}{dt} = g_{a_{i-1}}^p(t) - g_{a_i}^p(t) \qquad \forall p \in \mathcal{P}, i \in \{1, 2, ..., m(p)\} \tag{34}$$

$$x_{a_i}^p(t) = x_{a_{i,0}}^p \qquad \forall p \in \mathcal{P}, \quad i \in \{1, 2, ..., m(p)\} \tag{35}$$

$$h_p(t) = g^p_{a_1}(t + D_{a_1}(x_{a_1}(t)))(1 + D'_{a_1}(x_{a_1}(t))\dot{x}_{a_1}) \quad \forall p \in \mathcal{P} \tag{36}$$

$$g^p_{a_{i-1}}(t) = g^p_{a_i}(t + D_{a_i}(x_{a_i}(t)))(1 + D'_{a_i}(x_{a_i}(t))\dot{x}_{a_i}) \quad \forall p \in \mathcal{P}, i \in [2, m(p)] \tag{37}$$

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t)\, dt = Q_{ij} \quad \forall (i,j) \in \mathcal{W} \tag{38}$$

$$x^p_{a_i} \geq 0 \qquad g^p_{a_i} \geq 0 \qquad h_p \geq 0 \qquad \forall p \in \mathcal{P}, i \in \{1, 2, ..., m(p)\} \tag{39}$$

where Λ is the set of feasible controls (exit flows) defined previously. In the DUE constraints (33), we have introduced the notion of an *effective delay operator in the presence of tolls*, by which is meant

$$\Theta_p(t, x, y_p) = D_p(t, x) + F\{t + D_p(x, t) - T_A\} + y_p(t) \qquad \forall p \in \mathcal{P}$$

where y_p denotes the toll for path p. Of course we have the relationship

$$\Theta_p(t, x, y_p) = \Psi_p(t, x) + y_p(t) \tag{40}$$

where we recall from Friesz et al. (2001) that

$$y_p(t) = \sum_{i=1}^{m(p)} \delta_{a_i p} y_{a_i}\left(t + \Phi_{a_{i-1}}(t, x)\right) \quad \forall p \in \mathcal{P}$$

The variational-inequality constrained optimization problem (32) through (39) is a bi-level problem that is intrinsically difficult to solve. Note in particular that, even for a single instant of time, the number of constraints of the type (33) is uncountable.

In this paper, to numerically solve specific instances of (32)-(39), we may exploit the following alternative to expressing the underlying DUE problem as an infinite dimensional variation inequality:

Theorem 1 *Given that the effective travel delay for path p is $\Theta_p[t, x(t), y_p(t)]$, a nonnegative path flow vector $h \geq 0$ is a user equilibrium if and only if the conditions*

$$\Theta_p \geq \frac{\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Theta_p[t, x(t), y_p(t)]\, h_p(t)\, dt}{\sum_{p \in P_{ij}} \int_{t_0}^{t_f} h_p(t)\, dt} = \mu_{ij} \qquad \forall p \in \mathcal{P}_{ij}, \quad (i,j) \in \mathcal{W} \tag{41}$$

are satisfied

Proof: The dynamic user equilibrium condition stated in (13) and (14) can be modeled as an equivalent complementarity problem, that is

$$[\Theta_p(t, x^*) - \mu_{ij}]\, h^*_p(t) = 0, \qquad \Theta_p(t, x^*) - \mu_{ij} \geq 0, \qquad h^*_p(t) \geq 0 \tag{42}$$

for all $t \in [t_0, t_f], p \in \mathcal{P}_{ij}, (i,j) \in \mathcal{W}$. To show necessity we integrate the complementarity condition in (42) over the time horizon and summing for all paths, and obtain

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} [\Theta_p(t, x^*) - \mu_{ij}]\, h^*_p(t)\, dt = 0 \qquad \forall (i,j) \in \mathcal{W}$$

or

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Theta_p(t, x^*)\, h^*_p(t)\, dt = \mu_{ij} \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h^*_p(t)\, dt \qquad \forall (i,j) \in \mathcal{W} \tag{43}$$

To show sufficiency we re-state (41) as

$$\Theta_p - \mu_{ij} \geq \frac{\sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \Theta_p \left[t, x\left(t\right), y_p\left(t\right)\right] h_p\left(t\right) dt}{\sum_{p\in P_{ij}} \int_{t_0}^{t_f} h_p\left(t\right) dt} \qquad \forall p \in \mathcal{P}_{ij}, \quad (i,j) \in \mathcal{W} \tag{44}$$

and multiply both sides by path flow to obtain

$$0 = \left[\Theta_p\left(t, x^*\right) - \mu_{ij}\right] h_p^*\left(t\right) \geq \left[\frac{\sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \Theta_p \left[t, x\left(t\right), y_p\left(t\right)\right] h_p\left(t\right) dt}{\sum_{p\in P_{ij}} \int_{t_0}^{t_f} h_p\left(t\right) dt}\right] h_p^*\left(t\right) \qquad \forall p \in \mathcal{P}_{ij}, \quad (i,j) \in \mathcal{W} \tag{45}$$

from which (42) follows immediately.■

By virtue of Theorem 1, we may replace the DUE constraint (33) by the equality and inequality constraints (41) to obtain the following equivalent form of the DOTPEC:

$$\min \ J = \int_{t_0}^{t_f} \sum_{p\in\mathcal{P}} \Psi_p\left(t, x\right) h_p\left(t\right) dt \tag{46}$$

subject to

$$\mu_{ij} = \frac{\sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \Theta_p \left[t, x\left(t\right), y_p\left(t\right)\right] h_p\left(t\right) dt}{\sum_{p\in P_{ij}} \int_{t_0}^{t_f} h_p\left(t\right) dt} \qquad \forall \left(i,j\right) \in \mathcal{W} \tag{47}$$

$$\Theta_p \geq \mu_{ij} \qquad \forall p \in \mathcal{P}_{ij}, \quad (i,j) \in \mathcal{W} \tag{48}$$

$$\frac{dx_{a_i}^p\left(t\right)}{dt} = g_{a_{i-1}}^p\left(t\right) - g_{a_i}^p\left(t\right) \qquad \forall p \in \mathcal{P}, i \in \{1, 2, ..., m\left(p\right)\} \tag{49}$$

$$x_{a_i}^p\left(t\right) = x_{a_{i,0}}^p \qquad \forall p \in \mathcal{P}, \quad i \in \{1, 2, ..., m\left(p\right)\} \tag{50}$$

$$h_p\left(t\right) = g_{a_1}^p(t + D_{a_1}(x_{a_1}(t)))(1 + D_{a_1}'(x_{a_1}(t))\dot{x}_{a_1}) \quad \forall p \in \mathcal{P} \tag{51}$$

$$g_{a_{i-1}}^p\left(t\right) = g_{a_i}^p(t + D_{a_i}(x_{a_i}(t)))(1 + D_{a_i}'(x_{a_i}(t))\dot{x}_{a_i}) \quad \forall p \in \mathcal{P}, i \in [2, m(p)] \tag{52}$$

$$\sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p\left(t\right) dt = Q_{ij} \quad \forall \left(i,j\right) \in \mathcal{W} \tag{53}$$

$$x_{a_i}^p \geq 0 \qquad g_{a_i}^p \geq 0 \qquad h_p \geq 0 \qquad \forall p \in \mathcal{P}, i \in \{1, 2, ..., m\left(p\right)\} \tag{54}$$

Note that the above formulation is an infinite dimensional mathematical program with inequality and equality constraints in standard form, and that the number of constraints for any given instant of time is countable.

5 Multiple Time Scales

We have investigated the within-day behavior of road network users so far. In this section we describe a day-to-day adjust process that sets daily travel demand. Our perspective is is very simple: if today commuters experiences a level of congestion above a threshold representing the budget or tolerance for congestion of the typical commuter, travel demand will be less tomorrow and more workers will elect to stay at home (telecommute). To operationalize this idea, we take the perspective of evolutionary game theory to describe the day-to-day demand learning process in terms of the *moving average* of congestion and difference equations.

Let $\tau \in \Upsilon \equiv \{1, 2, ..., L\}$ be one typical discrete day within the planning horizon, and take the length of each day to be Δ, while the continuous clock time t within each day is presented by $t \in [(\tau - 1)\Delta, \tau\Delta]$ for all $\tau \in \{1, 2, ..., L\}$. The entire planning horizon spans L consecutive days. As noted above, we assume the travel demand for each day changes based on the moving average of congestion experienced over previous days. In fact we postulate that the travel demands Q_{ij}^{τ} for day τ between a given OD pair $(i, j) \in \mathcal{W}$ are determined by the following system of difference equations:

$$Q_{ij}^{\tau+1} = \left[Q_{ij}^{\tau} - s_{ij}^{\tau} \left\{ \frac{\sum_{p \in \mathcal{P}_{ij}} \sum_{j=0}^{\tau-1} \int_{j \cdot \Delta}^{(j+1)\cdot\Delta} \Psi_p \left[t, x\left(h^*, g^*\right)\right] dt}{|\mathcal{P}_{ij}| \cdot \tau \cdot \Delta} - \chi_{ij} \right\} \right]^{+} \quad \forall \tau \in \{1, 2, ...L - 1\} \tag{55}$$

with boundary condition

$$Q_{ij}^{1} = \tilde{Q}_{ij} \tag{56}$$

where $\tilde{Q}_{ij} \in \mathbb{R}_{+}$ is the fixed travel demand for the OD pair $(i, j) \in \mathcal{W}$ for the first day and χ_{ij} is the representative threshold. The operator $[x]^{+}$ is shorthand from $\max[0, x]$. The parameter s_{ij}^{τ} is related to the rate of change of inter-day travel demand.

6 Algorithms for Solving the DOTPEC

In this section, we provide two different algorithms for solving the DOTPEC: (1) descent in Hilbert space without time discretization, and (2) a finite dimensional discrete time approximation solved as a nonlinear program.

6.1 The Implicit Fixed Point Perspective

In both approaches, state-dependent time shifts must and can be accommodated using an implicit fixed point perspective, as innovated for the dynamic user equilibrium by Friesz and Mookherjee (2006). More specifically, in such an approach, one employs control and state information from a previous iteration to approximate current time shifted functions. This perspective may be summarized as follows:

1. Articulate the current approximate states (volumes) and controls (arc exit rates) by spline or other curve fitting techniques as continuous functions of time.

2. Using the aforementioned continuous functions of time, express time shifted controls as pure functions of time, while leaving unshifted controls as decision functions to be updated within the current iteration.

3. Update the states and controls, then repeat Step 2 and Step 3 until the control controls converge to a suitable approximate solution.

6.2 Descent in Hilbert Space

To articulate what is meant by descent in Hilbert space, it is much easier to study an abstract problem rather than the DOTPEC because of the notational complexity of the underlying DUE problem. To that end, let us consider an abstract optimal control problem with mixed state-control constraints involving state-dependent time shifts from the point of view of infinite dimensional mathematical programming:

$$\min \ J = \int_{t_0}^{t_f} F(x, u, u_D, t)dt \tag{57}$$

subject to

$$x(u, u_D, t) \in \Lambda = \left\{ x : \frac{dx}{dt} = f(x, u, u_D, t), x(0) = 0, G(x, u, u_D, t) = 0, x \geq 0 \right\} \in (\mathcal{H}^1[t_0, t_f])^n$$

where

$$\begin{aligned}
u &\in U \subseteq (L^2[t_0, t_f])^m \\
u_D &\equiv u(t + D(x)) : (\mathcal{H}^1[t_0, t_f])^n \times \mathbb{R}^1_+ \longrightarrow (L^2[t_0, t_f])^m \\
f &: (\mathcal{H}^1[t_0, t_f])^n \times (L^2[t_0, t_f])^{2m} \times \mathbb{R}^1_+ \longrightarrow (L^2[t_0, t_f])^m \\
F &: (\mathcal{H}^1[t_0, t_f])^n \times (L^2[t_0, t_f])^{2m} \times \mathbb{R}^1_+ \longrightarrow (L^2[t_0, t_f])^m \\
G &: (\mathcal{H}^1[t_0, t_f])^n \times (L^2[t_0, t_f])^{2m} \times \mathbb{R}^1_+ \longrightarrow (L^2[t_0, t_f])^m
\end{aligned}$$

In the above, $(L^2[t_0, t_f])^m$ is the m-fold product of the space of square integrable functions $L^2[t_0, t_f]$ and $(\mathcal{H}^1[t_0, t_f])^n$ is the n-fold product of the Sobolev space $\mathcal{H}^1[t_0, t_f]$ for the real interval $[t_0, t_f] \subset \mathbb{R}^1_+$. In applying descent in Hilbert space to this problem, it is convenient to use quadratic-loss penalty functions and a logarithmic barrier function to create the unconstrained program:

$$\min \ J_1 = \int_{t_0}^{t_f} F(x, u, u_D, t)dt + \frac{1}{2}\int_{t_0}^{t_f} \sum_i \eta_i (G_i(x, u, u_D, t))^2 dt + \frac{1}{2}\int_{t_0}^{t_f} \sum_i \rho_i \min(0, x_i)^2 dt \tag{58}$$

where it is understood that x denotes the operator

$$x(u, u_D, t) \in \Lambda_1 = \left\{ x : \frac{dx}{dt} = f(x, u, u_D, t), x(0) = x_0 \right\} \in (\mathcal{H}^1[t_0, t_f])^n,$$

and η_i and ρ_i are penalty and barrier multipliers to be adjusted from iteration to iteration. The resulting problem can be solved using a continuous time steepest descent method. For the penalized criterion (), the algorithm can be stated as following:

Step 0. **Initialization**. Pick $u^0(t) \in U$ and set $k = 1$.
Step 1. **Finding state variables**. Solve the state dynamics

$$\begin{aligned}
\frac{dx}{dt} &= f(x, u^{k-1}, u_D^{k-1}, t) \\
x(0) &= x_0
\end{aligned}$$

and call the solution $x^k(t)$, using curve fitting to create an approximation to $x^k(t)$ when necessary.
Step 2. **Finding adjoint variables**. Solve the adjoint dynamics

$$\begin{aligned}
-\frac{d\lambda}{dt} &= \left[\nabla_x H(x, u^{k-1}, u_D^{k-1}, \lambda, t) \right]_{x=x^k} \\
\lambda(t_f) &= 0
\end{aligned}$$

where the Hamiltonian is given by

$$H(x,u,u_D,\lambda,t) = F(x,u,u_D,t) + \frac{1}{2}\sum_i \rho_i \min(0,x_i)^2 + \frac{1}{2}\sum_i \eta_i (G_i(x,u,u_D,t))^2 + \lambda^T f(x,u,u_D,t)$$

Call the solution $\lambda^k(t)$, using curve fitting to create an approximation to $\lambda^k(t)$ when necessary..
Step 3. **Finding the gradient**. Determine

$$\nabla_u J^k \equiv \left[\nabla_u H(x^k,u,u_D^{k-1},\lambda^k,t)\right]_{u=u^k}$$

Step 4. **Updating the current control**. For a suitably small step size

$$\theta_k \in \mathbb{R}^1_{++}$$

determine

$$u^k(t) = u^{k-1}(t) - \theta_k \nabla_u J^k$$

Step 5. **Stopping Test**. For $\epsilon \in \mathbb{R}^1_{++}$, a preset tolerance, stop if

$$||u^{k+1} - u^k|| < \epsilon$$

and declare

$$u^* \approx u^{k+1}$$

Otherwise set $k = k+1$ and go to Step1.

6.3 Discrete-time Approximation of DOTPEC

The optimal control problem (46)-(54) may be given the following discrete time approximation:

$$\min \ J = \sum_{k=0}^{N} \sum_{p\in\mathcal{P}} \phi(k)\, \Psi_p\left[t_k, x(t_k)\right] h_p(t_k)\, \Delta$$

subject to

$$\mu_{ij} = \frac{\sum_{p\in\mathcal{P}_{ij}} \sum_{k=0}^{N} \phi(k)\, \Theta_p\left[t_k, x(t_k), y_p(t_k)\right] h_p(t_k)\, \Delta}{\sum_{p\in P_{ij}} \sum_{k=0}^{N} \phi(k)\, h_p(t_k)\, \Delta} \qquad \forall (i,j) \in \mathcal{W}$$

$$\Theta_p(t_k) \geq \mu_{ij} \qquad \forall k \in [0,N], \quad p \in \mathcal{P}_{ij}, \quad (i,j) \in \mathcal{W}$$

$$x^p_{a_i}(t_{k+1}) = x^p_{a_i}(t_k) + \Delta\left[g^p_{a_{i-1}}(t_k) - g^p_{a_i}(t_k)\right]$$
$$\forall k \in [0,N-1], \quad p \in \mathcal{P}, \quad i \in [1,m(p)]$$

$$x^p_{a_i}(t_0) = x^p_{a_{i,0}} \qquad \forall p \in \mathcal{P}, \quad i \in \{1,2,...,m(p)\}$$

$$x(t_k) \geq 0 \qquad \forall k \in [0,N]$$

$$h_p(t_k) = g^p_{a_1}(t_k + D_{a_1}(x_{a_1}(t_k)))(1 + D'_{a_1}(x_{a_1}(t_k))\dot{x}_{a_1}) \quad \forall k \in [0,N], p \in \mathcal{P}$$

$$g^p_{a_{i-1}}(t_k) = g^p_{a_i}(t_k + D_{a_i}(x_{a_i}(t_k)))(1 + D'_{a_i}(x_{a_i}(t_k))\dot{x}_{a_i})$$
$$\forall k \in [0,N], p \in \mathcal{P}, i \in [2,m(p)]$$

$$\sum_{p\in\mathcal{P}_{ij}} \sum_{k=0}^{N} \phi(k)\, h_p(t_k)\, \Delta = Q_{ij} \quad \forall (i,j) \in \mathcal{W}$$

$$y_a(t_k) \geq 0 \qquad \forall a \in \mathcal{A}, \quad k \in [0,N]$$

$$x(t_k) \geq 0 \qquad g(t_k) \geq 0 \qquad h(t_k) \geq 0 \qquad \forall k \in [0,N]$$

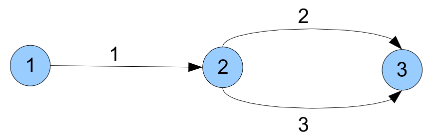

Figure 1: 3-Arc 3-Node Traffic Network

where k takes non-negative integer values, Δ is the discrete time step that divides the time interval $[t_0, t_f]$ into N equal segments, $\phi(k)$ is the coefficient which arises from a trapezoidal approximation of integrals, that is

$$\phi(k) = \begin{cases} 0.5 & \text{if } k = 0 \text{ and } N \\ 1 & \text{otherwise} \end{cases}$$

and

$$t_k = k\Delta$$

One advantage of time discretization is that we can now completely eliminate state variables (arc volumes) from the problem by noting that

$$x_{a_i}^p(t_{k+1}) = x_{a_{i,0}}^p + \sum_{r=0}^{k} \Delta \left[g_{a_{i-1}}^p(t_r) - g_{a_i}^p(t_r) \right] \quad \forall k \in [0, N-1], \quad p \in \mathcal{P}, \quad i \in \{1, 2, ..., m(p)\}$$

As a consequence, one obtains a finite dimensional mathematical program, which may be solved by conventional algorithms developed for such problems. We employ GAMS/MINOS for the numerical example of Section 7.1.

7 Numerical Example

In what follows, we consider a 3 arc, 3 node network shown in Figure 1. The arc labels and arc delay functions for this network are summarized in the following table:

Arc name	From node	To node	Arc delay, $D_a(x_a(t))$
a_1	1	2	$2 + (x_{a_1}/200)$
a_2	2	3	$1 + (x_{a_2}/150)$
a_3	2	3	$3 + (x_{a_3}/100)$

There are 2 paths connecting the single OD pair formed by nodes 1 and 3, namely:

$$\mathcal{P}_{13} = \{p_1, p_2\}, \qquad p_1 = \{a_1, a_2\}, \qquad p_2 = \{a_1, a_3\}$$

The controls (path flows and arc exit flows) and states (path-specific arc traffic volumes) associated with the network are:

Path	Path Flow	Arc Exit Flow	Traffic Volume of Arc
p_1	h_{p_1}	$g_{a_1}^{p_1}, g_{a_2}^{p_1}$	$x_{a_1}^{p_1}, x_{a_2}^{p_1}$
p_2	h_{p_2}	$g_{a_1}^{p_2}, g_{a_3}^{p_2}$	$x_{a_1}^{p_2}, x_{a_3}^{p_2}$

We consider three-day toll planning in which each day is 24 hours, hence, $\Delta = 24$ and $L = 14$ (two weeks). We assume there is the initial travel demand $\tilde{Q} = 150$ units from node 1 (origin) to

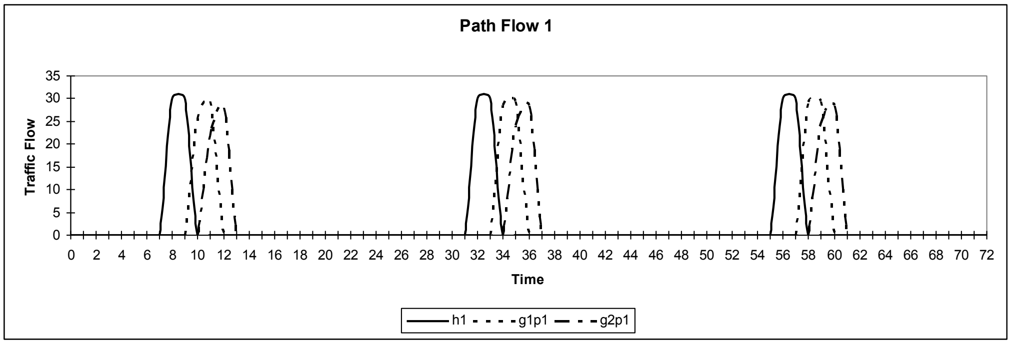

Figure 2: Path and arc exit flows for path p_1.

node 3 (destination). The threshold for travel cost is $\chi = 20000$ and the inter-day rate of change in travel demand is $s_{13} = 0.7$. The desired arrival time for each day is $t_A = 12$, and we employ the symmetric early/late arrival penalty

$$F\left[t + D_p(x,t) - t_A\right] = 5\left[t + D_p(x,t) - t_A\right]^2$$

Further, without any loss of generality, we take

$$x_{a_i}^p(0) = 0 \quad \forall i \in [1, m(p)], p \in \mathcal{P}$$

In what follows we forgo the detailed symbolic statement of this example, and, instead, provide numerical results in graphical form.

7.1 DOTPEC Computation Based on Time Discretization and GAMS/ MINOS

Path flows and arc exit flows for paths p_1 and p_2 are presented in Figures 2 and 3, while path flows and tolls for each arc are given in Figures 4, 5 and 6, for three days from the computed fourteen-day results. We see that tolls tend to be proportional to the path flows. When, for path p_1, we compare the effective path delays (including tolls) with path flows (origin departure rates) by plotting both for the same time scale, Figure 7 is obtained. This figure shows that departure rate peaks when the associated effective path delay achieves a local minimum, thereby demonstrating that a dynamic user equilibrium has been found. Similar comparisons are made for paths p_2 in Figure 8. The daily changes of travel demand from the origin to destination according to the difference equation (55) are given in Figure 9.

7.2 DOTPEC Computation based on Descent in Hilbert Space

The same numerical example was also solved by descent in Hilbert space, a continuous-time numerical scheme described in Section 6.2. While employing the implicit fixed point approach, we penalize the flow propagation constraints, the travel demand constraint, and the DUE conditions which are converted to a set of inequality constriants. We present the path tolls in Figures 10 and 11. As in the previous section we again show the resulting flows are a dynamic user equilibrium by plotting the travel cost and departure flow on the same time axis in Figures 12 and 13.

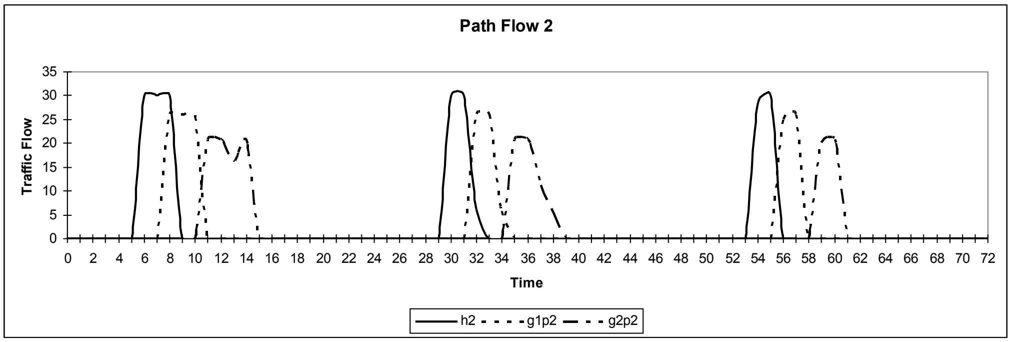

Figure 3: Path and arc exit flows for path p_2.

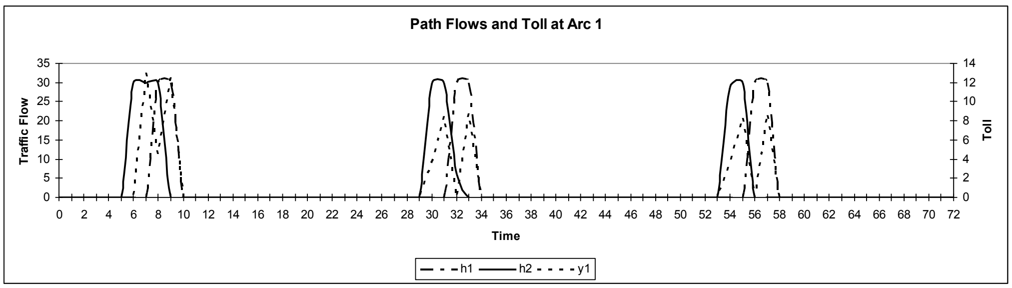

Figure 4: Path flows and toll at arc a_1.

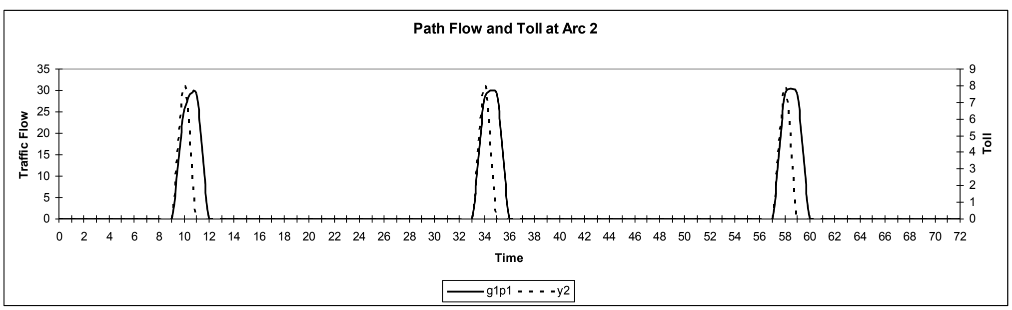

Figure 5: Path flow and toll at arc a_2.

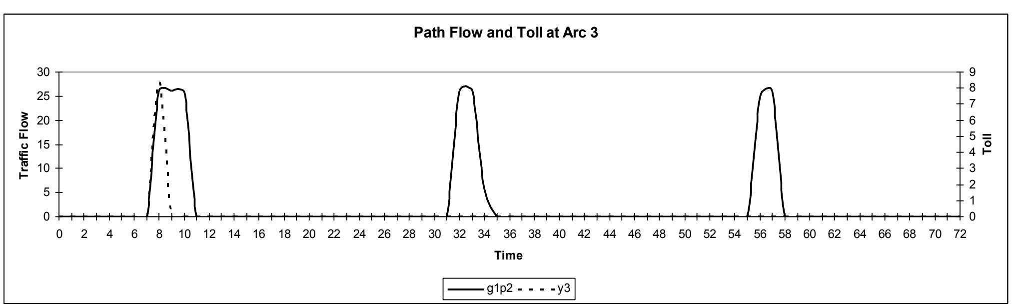

Figure 6: Path flow and toll at arc a_3.

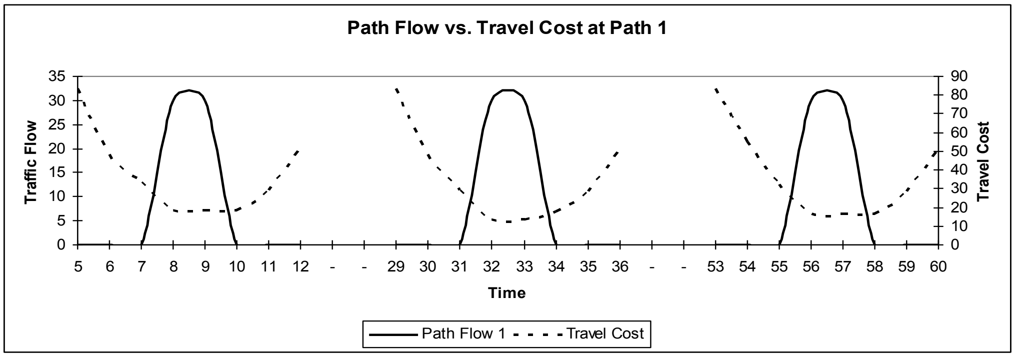

Figure 7: Comparison of path flow and associated unit travel costs for path p_1.

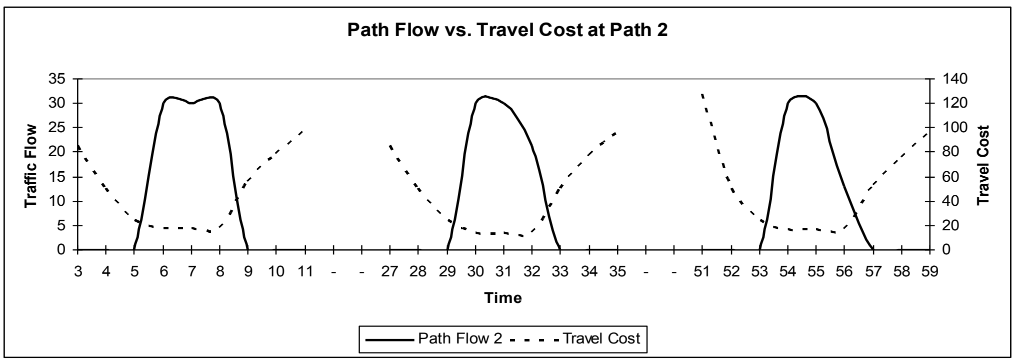

Figure 8: Comparison of path flow and associated unit travel costs for path p_2.

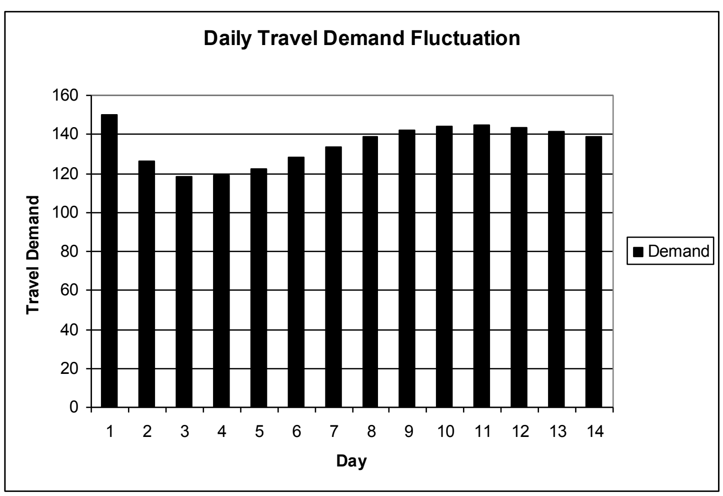

Figure 9: Daily changes of travel demand from the origin (node 1) to the destination (node 3)

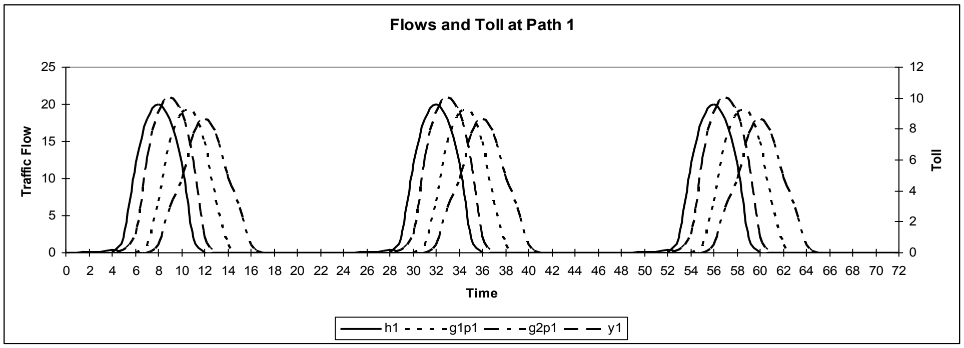

Figure 10: Path flows and toll at path p_1.

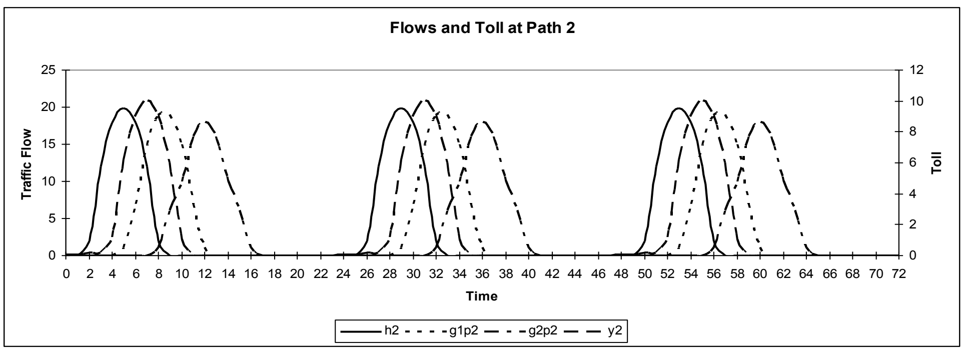

Figure 11: Path flows and toll at path p_2.

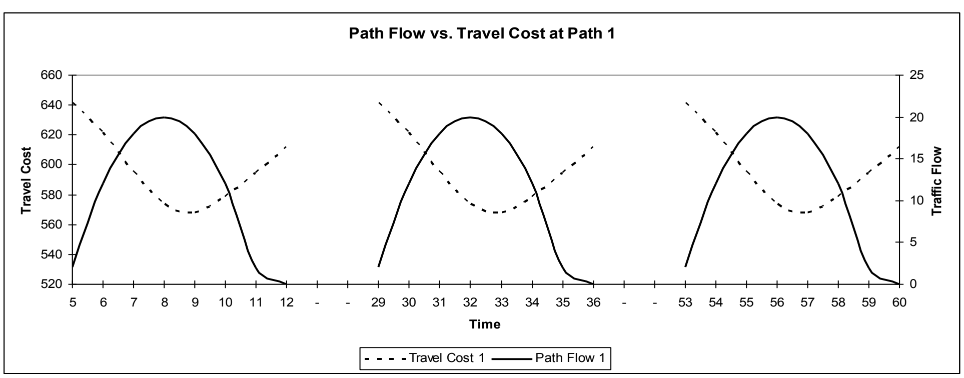

Figure 12: Comparison of path flow and associated unit travel costs for path p_1.

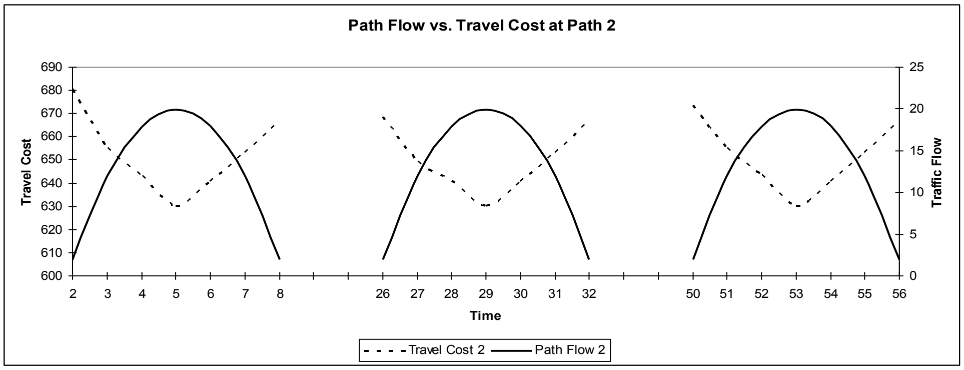

Figure 13: Comparison of path flow and associated unit travel costs for path p_2.

7.3 Comparison of Tolls

To compare, the tolls by DETP and DOTPEC with two algorithms of choice, we suggest a computational scheme for DETP. Recall that the decision rule for the dynamic efficient toll is:

$$y_p^U(t) = \frac{\partial \Psi_p\left(t, x^S\right)}{\partial h_p} h_p^S \qquad \forall t \in [t_0, t_f]$$

Note that the partial derivative of $\Psi_p\left(t, x^S\right)$ with respect to the path flow h_p is not zero, since the state variable x is an implicit function of the control h_p as the relationship is expressed in the state dynamics. Further we cannot calculate the derivative directly due to the nested delay operator appears in $\Psi_p(\cdot, \cdot)$. However, from the numerical study of the dynamic system optimum traffic assignment, it is known that the controls are zero or singular. When the departure rate is nonzero, it as well as the states obtained from it are smooth and the delay operator is differentiable, although the derivative $\frac{\partial \Psi_p\left(t, x^S\right)}{\partial h_p}$ does not exist at the time moments where there are kinks in the controls. The derivative is numerically approximated as:

$$\frac{\partial \Psi_p\left[t, x\left(h^*, g^*\right)\right]}{\partial h_p} \cong \frac{\Psi_p\left[t, x\left(h+\delta, g\right)\right] - \Psi_p\left[t, x\left(h, g\right)\right]}{\delta}$$

A numerical comparison of the tolls found from the DETP with those from the DOTPEC is given in Figure Figures 14 and 15. We see that the efficient toll has a more spike-like behavior than that for the DOTPEC. It is also interesting to note that the total congestion cost for the DETP is $(26.43, 38.85)$ while the total congestion cost for the DOTPEC is $(38.30, 46.85)$ by discrete approximation and $(43.09, 45.13)$ by descent in Hilbert spaces for paths (p_1, p_2).

8 Concluding Remarks

We have presented a mathematical formulation of the DOTPEC and have shown how it may be directly solved using the notion of descent in Hilbert space for a small illustrative problem. We have also computed solutions using the more familiar approach of time discretization combined with off-the-shelf nonlinear programming software. Clearly, in-depth testing and comparison of these solution methods is required before one can be recommended over the other.

We have not explored in this manuscript the difficult theoretical questions of algorithm convergence, existence of solutions to the dynamic efficient toll and the DOTPEC problems, the Braess

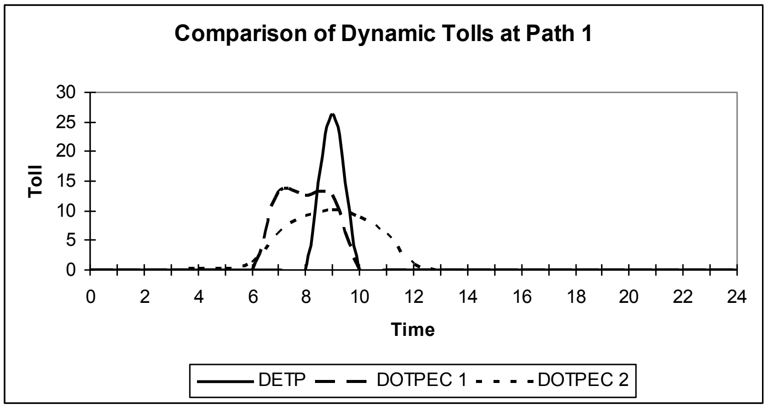

Figure 14: Comparison of Dynamic Tolls by DETP, DOTPEC solved by discrete time approximation (DOTPEC 1), and DOTPEC solved by descent in Hilbert spaces (DOTPEC 2) for path p_1

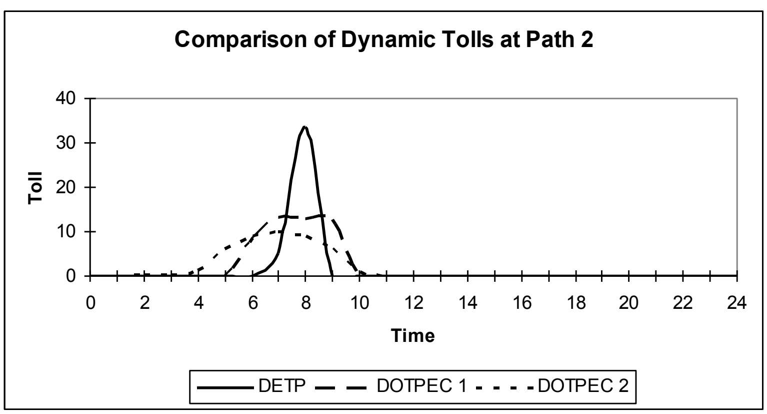

Figure 15: Comparison of Dynamic Tolls by DETP, DOTPEC solved by discrete time approximation (DOTPEC 1), and DOTPEC solved by descent in Hilbert spaces (DOTPEC 2) for path p_2

paradox and the price of anarchy. These topics are being addressed in a separate manuscript still in preparation. Given that serious efforts are already under way to implement versions of the optimal dynamic toll problem in the U.S. and elsewhere, our initial focus on computation seems fully justified.

We close by commenting that analytical DUE models – in our opinion – are far and away the best starting point for studies of the theoretical aspects of dynamic efficient tolls and dynamic congestion pricing. In particular, we have shown in this paper that an intuitive generalization to a dynamic setting of the efficient static toll rule is correct – something that could not be established in such a definitive way with a simulation model.

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