Computing Dual Time Scale Dynamic User Equilibria

Terry L. Friesz   Tae Il Kim   Changhyun Kwon
Matthew A. Rigdon

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Abstract

In this paper we summarize a dual-time scale formulation of dynamic user equilibrium with demand growth due to Friesz et al (2008). This formulation belongs to the problem class that Pang and Stewart (2008) refer to as *differential variational inequalities*. We also present a fixed point algorithm for computing solutions to the dual time-scale model without calculating derivatives, along with a numerical example.

Keywords: dynamic user equilibrium, differential variational inequalities, fixed point algorithm, dual time scale

1 Introduction

This paper presents in summary form one type of dynamic traffic assignment known as *dynamic user equilibrium* (DUE). Our model recognizes tactical routing and departure time decisions are made in continuous time (the within-day time scale) while demand evolves in discrete time (the day-to-day time scale) and that the two time scales are coupled. Our dual time-scale formulation of dynamic user equilibrium with demand growth belongs to the problem class known as *differential variational inequalities*, according to terminology introduced by Pang and Stewart (2008).

Friesz et al. (1993) introduced the notion of exit time functions together with a variational inequality to describe dynamic user equilibrium; that model is consistent with FIFO for appropriate arc delay functions, even though explicit flow propagation constraints are not employed. In particular, they introduce a function $\xi^p_{a_i}(t)$ that expresses the time of exit from arc a_i of every path

$$p = \left\{a_1, a_2, ..., a_{i-1}, a_i, a_{i+1}, ..., a_{m(p)}\right\} \in \mathcal{P} \ , \tag{1}$$

where $\mathcal{P}$ is the set of all network paths. The exit time functions obey the recursive relationships

$$\xi^p_{a_1} = t + D_{a_1}\left[x_{a_1}(t)\right] \qquad \forall \ p \in \mathcal{P} \tag{2}$$

$$\xi^p_{a_i} = \xi^p_{a_{i-1}}(t) + D_{a_i}\left[x_{a_i}\left(\xi^p_{a_{i-1}}(t)\right)\right] \qquad \forall\, p \in \mathcal{P},\ i \in [2, m(p)] \tag{3}$$

where $D_{a_i}[x_{a_i}(t)]$ is the time to traverse arc a_i; it is a function of the number of vehicles x_{a_i} in front of the entering vehicle at the time of entry. This model of arc delay is frequently called the point queue model. The arc exit time may be used to express the path delay

$$D_p = \sum_{i=1}^{m(p)} \xi^p_{a_i} = \xi^p_{a_m} - t$$

Friesz et al. (1993) also gave the first continuous time articulation of flow conservation based on a fixed within-day trip matrix:

$$\sum_{p \in \mathcal{P}_{ij}} \int_0^T h_p dt = Q_{ij} \qquad \forall\,(i,j) \in \mathcal{W} \tag{4}$$

where $\mathcal{W}$ is the set of all origin-destination pairs, $\mathcal{P}_{ij}$ is the set of paths connecting $(i,j) \in \mathcal{W}$ and h_p is the departure rate from the origin of path $p \in \mathcal{P}_{ij}$, while Q_{ij} is the travel demand between $(i,j) \in \mathcal{W}$ and $[0,T] \in \Re^1_+$ is the continuous time interval representing a single day or commuting period of interest. They used (2) and (3) together with dynamics expressed as integral equations involving inverse exit time functions to define an effective path delay operator. That operator, in turn, was used with (4) and non-negativity restrictions to construct an infinite dimensional variational inequality whose solutions are dynamic user equilibria; their formulation is the first expression of dynamic user equilibrium as a a variational inequality. Subsequently, Wu et al. (1998) and Xu et al. (1999) developed algorithms for the Friesz et al. (1993) model. In particular they studied the use of the projected gradient method and solved some modest size test problems, but did not provide useful convergence results. Zhu and Marcotte (2000) prove the existence of solutions to the Friesz et al. (1993) model when departure rates are stipulated to be bounded from above. More recently, Bliemer and Bovy (2003) have extended the Friesz et al. (1993) formulation by introducing multiple user classes, thereby creating a quasi-variational inequality.

Friesz et al. (2001) employed path delays computed from (2) and (3) with dynamics

$$\frac{dx^p_{a_1}(t)}{dt} = h_p(t) - g^p_{a_1}(t) \qquad \forall p \in \mathcal{P} \tag{5}$$

$$\frac{dx^p_{a_i}(t)}{dt} = g^p_{a_{i-1}}(t) - g^p_{a_i}(t) \qquad \forall\, p \in \mathcal{P}, i \in [2, m(p)]\ , \tag{6}$$

where $x^p_{a_i}$ is the volume of traffic on arc a_i of path p for $i \in [1, m(p)]$ and $g^p_{a_i}(t)$ denotes the flow exiting that same arc, to formulate the dynamic user equilibrium problem as a differential variational inequality that is completely equivalent to the Friesz et al. (1993) infinite dimensional variational inequality formulation. Friesz et al. (2001) included in their formulation the flow propagation constraints

$$g_{a_1}\left(t + D_{a_1}\left[x_{a_1}(t)\right]\right)\left(1 + D'_{a_1}\left[x_{a_1}(t)\right]\dot{x}_{a_1}\right) = h_p(t) \tag{7}$$

$$g^p_{a_i}\left(t + D_{a_i}\left[x_{a_i}(t)\right]\right)\left(1 + D'_{a_i}\left[x_{a_i}(t)\right]\dot{x}_{a_i}(t)\right) = g^p_{a_{i-1}}(t)$$
$$\forall\ p \in P,\ \ i \in [2, m(p)] \tag{8}$$

which are identical to those proposed by Astarita (1995) and which include consideration of expanding/contracting platoons of vehicles. Friesz and Mookherjee (2006) propose and test a fixed point algorithm implemented in continuous time to solve the differential variational inequality formulation of Friesz et al. (2001); that algorithm requires monotonic path delay operators to assure convergence and, hence, is a heuristic in practice.

The paper by Li et al. (2000) is one of several that uses the Friesz et al. (1993) recursive equations (2) and (3) that are based on exit time functions along with the flow propagation constraints (7) and (8) to express path delay and assure physically meaningful flow. They express the DUE conditions in discrete time and show it is equivalent to a finite dimensional variational inequality. They offer an ad hoc algorithm without discussing convergence.

2 The Within-Day Differential Variational Inequality Formulation

We will, for the time being, assume the time interval of analysis is a single commuting period

$$[t_0, t_f] \subset \Re^1_+$$

where $t_f > t_0$. The most crucial ingredient of a dynamic user equilibrium model is the path delay operator, which provides the delay on any path p per unit of flow departing from the origin of that path; it is denoted by

$$D_p(t,h) \qquad \forall p \in \mathcal{P} \tag{9}$$

where $\mathcal{P}$ is the set of all paths employed by travelers, t denotes departure time, and h is a vector of departure rates. From these we construct effective unit path delay operators $\Psi_p(t,h)$ by adding the so-called schedule delay $F\left[t + D_p(t,h) - T_A\right]$; that is

$$\Psi_p(t,h) = D_p(t,h) + F\left[t + D_p(t,h) - T_A\right] \qquad \forall p \in P \tag{10}$$

where T_A is the desired arrival time and $T_A < t_f$. The function $F(\cdot)$ assesses a penalty whenever

$$t + D_p(t,h) \neq T_A \tag{11}$$

since $t + D_p(t,h)$ is the clock time at which departing traffic arrives at the destination of path $p \in \mathcal{P}$. The path delay operators may be obtained from an embedded delay model, data combined with response surface methodology, or data combined with inverse modeling. Unfortunately, regardless of how derived, realistic path delay operators do not possess the desirable property of monotonicity; they may also be non-differentiable. We will have more to say about path delays when we discuss dynamic network loading in Section 4.

For the time being, there will be a fixed trip matrix

$$Q = (Q_{ij} : (i,j) \in \mathcal{W})$$

where each $Q_{ij} \in \Re^1_{++}$ is the fixed travel demand, expressed as a volume, between origin-destination pair $(i,j) \in \mathcal{W}$ and $\mathcal{W}$ is the set of all origin-destination pairs. Additionally, we will define the set $\mathcal{P}_{ij}$ to be the subset of paths that connect origin-destination pair $(i,j) \in \mathcal{W}$. We denote the space of square integrable functions for the real interval $[t_0, t_f]$ by $L^2[t_0, t_f]$. We stipulate that

$$h \in \left(L^2_+[t_0, t_f]\right)^{|\mathcal{P}|}$$

We write the flow conservation constraints as

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t)\, dt = Q_{ij} \qquad \forall (i,j) \in \mathcal{W} \tag{12}$$

where (12) is comprised of Lebesgue integrals. We define the set of feasible flows by

$$\Lambda_0 = \left\{ h \geq 0 : \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t)\, dt = Q_{ij} \qquad \forall (i,j) \in \mathcal{W} \right\} \subseteq \left(L^2_+[t_0, t_f]\right)^{|\mathcal{P}|} \tag{13}$$

Let us also define the infimum of effective travel delays

$$v_{ij} = ess \inf \left[\Psi_p(t,h) : p \in \mathcal{P}_{ij}\right] \qquad \forall (i,j) \in \mathcal{W} \tag{14}$$

We now offer the following definition of dynamic user equilibrium first articulated by Friesz et al. (1993):

Definition 1 *Dynamic user equilibrium. A vector of departure rates (path flows) $h^* \in \Lambda_0$ is a dynamic user equilibrium if*

$$h_p^*(t) > 0, p \in P_{ij} \Longrightarrow \Psi_p[t, h^*(t)] = v_{ij} \tag{15}$$

We denote this equilibrium by $DUE(\Psi, \Lambda_0, [t_0, t_f])$.

The meaning of Definition 1 is clear: positive departure rates at a particular time along a particular path must coincide with least effective travel delay. An implication of the definition is that

$$\Psi_p(t, h^*) > v_{ij}, p \in P_{ij} \Longrightarrow h_p^* = 0 \tag{16}$$

Using measure theoretic arguments, Friesz et al. (1993) established that a dynamic user equilibrium is equivalent to the following variational inequality under suitable regularity conditions:

$$\left.\begin{array}{c}\text{find } h^{*} \in \Lambda_0 \text{ such that} \\ \sum_{p\in\mathcal{P}} \displaystyle\int_{t_0}^{t_f} \Psi_p(t,h^{*})(h_p - h_p)dt \geq 0 \\ \forall h \in \Lambda_0 \end{array}\right\} \tag{17}$$

It is not widely understood, however, that (17) is equivalent to a *differential variational inequality.* This is formally established in Friesz et al. (2008) who note that the flow conservation constraints may be replaced by a two point boundary value problem. In particular, (17) may be expressed as

$$\left.\begin{array}{c}\text{find } h^{*} \in \Lambda \text{ such that} \\ \sum_{p\in\mathcal{P}} \displaystyle\int_{t_0}^{t_f} \Psi_p(t,h^{*})(h_p - h_p)dt \geq 0 \\ \forall h \in \Lambda \end{array}\right\} DVI(\Psi,\Lambda,[t_0,t_f]) \tag{18}$$

where

$$\Lambda = \left\{ h \geq 0 : \frac{dy_{ij}}{dt} = \sum_{p\in P_{ij}} h_p\left(t\right),\ y_{ij}(t_0) = 0,\ y_{ij}\left(t_f\right) = Q_{ij} \quad \forall \left(i,j\right) \in \mathcal{W} \right\} \tag{19}$$

Using appropriate optimality conditions from the theory of optimal control to analyze differential variational inequality (18) and (19), Friesz et al. (2008) establish the following result:

Theorem 1 *Dynamic user equilibrium equivalent to a differential variational inequality. Assume* $\Psi_p(\cdot,h) : [t_0,t_f] \longrightarrow \Re^1_+$ *is measurable and strictly positive for all* $p \in \mathcal{P}$ *and all* $h \in \Lambda$*. A vector of departure rates (path flows)* $h^* \in \Lambda$ *is a dynamic user equilibrium if and only if* h^* *solves* $DVI(\Psi,\Lambda,[t_0,t_f])$*, as defined by (18).*

3 The Dual Time Scale Model

Let

$$\tau \in \Upsilon \equiv \{1,2,...,N\}$$

be one typical discrete day and take the length of each day to be Δ, while the continuous clock time within each day reads $t \in [(\tau-1)\Delta, \tau\Delta]$ for all $\tau \in \{1,2,...,N\}$. The entire planning horizon spans N consecutive days. Let us suppose we have a demand growth model of the abstract form

$$\begin{aligned} Q_{ij}^{\tau+1} &= \mathcal{F}_{ij}\left(Q_{ij}^{\tau}, h^1, h^2, ..., h^{\tau}; \theta\right) \\ & \qquad \forall \left(i,j\right) \in \mathcal{W},\ \tau \in \{1,2,...N-1\} \end{aligned} \tag{20}$$

$$
\begin{aligned}
Q_{ij}^{\tau} &\geq 0 \quad \forall (i,j) \in \mathcal{W},\ \tau \in \{2,...,N\} &(21)\\
Q_{ij}^{1} &= K_{ij} \in \Re_{+}^{1} \quad \forall (i,j) \in \mathcal{W} &(22)
\end{aligned}
$$

where

$$
\begin{aligned}
Q_{ij}^{\tau} &= \text{origin-destination travel demand } \forall (i,j) \in \mathcal{W},\ \tau \in \{1,2,...,N\}\\
Q^{\tau} &= \left(Q_{ij}^{\tau} : (i,j) \in \mathcal{W}\right)\\
Q &= (Q^{\tau} : \tau \in \Upsilon)\\
K_{ij} &= \text{a known, non-negative constant } \forall (i,j) \in \mathcal{W}\\
h^{\tau} &= \left(h_p^{\tau} : p \in \mathcal{P}\right)\\
h &= (h^{\tau} : \tau \in \Upsilon)\\
\theta &= \text{a vector of model parameters}
\end{aligned}
$$

Note the change in notation: now h is a tuple of daily flow vectors h^{τ} rather than merely a vector of flows for one representative day. Also we define

$$
\Lambda_{\tau}^{0}(Q^{\tau}) = \left\{ h^{\tau} \geq 0 : \sum_{p \in \mathcal{P}_{ij}} \int_{(\tau-1)\cdot\Delta}^{\tau\cdot\Delta} h_p^{\tau}(t)\, dt = Q_{ij}^{\tau} \quad \forall (i,j) \in \mathcal{W} \right\}
$$

which is of course equivalent to

$\Lambda_{\tau}(Q^{\tau}) =$

$$
\left\{ \begin{array}{r} h^{\tau} \geq 0 : dy_{ij}/dt = \sum_{p \in \mathcal{P}_{ij}} h_p(t),\ y_{ij}\left[(\tau-1)\cdot\Delta\right] = 0,\ y_{ij}(\tau\cdot\Delta) = Q_{ij}^{\tau} \\ \forall (i,j) \in \mathcal{W} \end{array} \right\}
$$

We note that

$$
\Lambda_{\tau}(Q^{\tau}) \in \left(L^2\left[\tau\cdot\Delta, (\tau-1)\cdot\Delta\right]\right)^{|\mathcal{P}|}
$$

and also define

$$
\Lambda(Q) = \prod_{\tau=1}^{N} \Lambda_{\tau}(Q^{\tau}) \tag{23}
$$

A dual time scale model of dynamic user equilibrium with endogenous demand growth is

$$
\left.
\begin{array}{c}
\text{find } Q^* \geq 0 \text{ and } h^* \in \Lambda(Q^*) \text{ such that} \\[2ex]
\sum_{p \in \mathcal{P}} \int_{(\tau-1)\cdot\Delta}^{\tau\cdot\Delta} \Psi_p(t, h^{\tau*}) \left(h_p^{\tau} - h_p^{\tau*}\right) dt \geq 0 \quad \forall \tau \in \Upsilon, h^{\tau} \in \Lambda_{\tau}(Q^{\tau*}) \\[2ex]
Q_{ij}^{\tau+1,*} = \mathcal{F}_{ij}\left(Q_{ij}^{\tau*}, h^{\tau*}, h^{\tau-1,*}, ..., h^{1*}; \theta\right) \quad \forall (i,j) \in \mathcal{W}, \tau \in \{1,2,...,N-1\} \\[2ex]
Q_{ij}^{1*} = K_{ij} \quad \forall (i,j) \in \mathcal{W}
\end{array}
\right\} \tag{24}
$$

The dual time scale model (24) may be solved by time stepping, so that exactly one continuous time variational inequality is faced for each day τ. To understand why time-stepping works for (24), note that when $\tau = 1$ we know each $Q_{ij}^{1*} = K_{ij}$ so that we also know

$$Q^{1*} = (K_{ij} : (i,j) \in \mathcal{W})$$

Thus, we face the well-defined problem of finding $h^{*1} \in \Lambda_1 \left(Q^{1*}\right)$ such that

$$\sum_{p \in \mathcal{P}} \int_0^{\Delta} \Psi_p(t, h^{1*}) \left(h_p^1 - h_p^{*1}\right) dt \geq 0 \quad \forall h^1 \in \Lambda_1 \left(Q^{1*}\right) \tag{25}$$

The solution of (25) allows us, using the day-to-day demand dynamics, to compute

$$Q_{ij}^{2*} = \mathcal{F}_{ij} \left(Q_{ij}^{1*}, h^{1*}; \theta\right) \quad \forall (i,j) \in \mathcal{W} \tag{26}$$

and thereby determine the vector Q^{2*}, setting the stage for solving the next within-day differential variational inequality to find h^{2*}. This process, known as time-stepping, leads us eventually to a complete solution of (24). It also focuses attention on the need for an algorithm to solve the continuous time differential variational inequality faced for each value of τ.

As an example of dynamics governing the evolution of travel demand, one may postulate that, for each day τ, the travel demands Q_{ij}^{τ} between each given origin-destination pair are determined by the following system of difference equations:

$$Q_{ij}^{\tau+1} = \left[Q_{ij}^{\tau} - s_{ij}^{\tau} \left\{ \frac{\sum_{p \in \mathcal{P}_{ij}} \sum_{j=1}^{\tau} \int_{(j-1)\cdot\Delta}^{j\cdot\Delta} \Psi_p\left(t, h^{j*}\right) dt}{|\mathcal{P}_{ij}| \cdot \tau \cdot \Delta} - \chi_{ij} \right\} \right]^+ \quad \forall (i,j) \in \mathcal{W}, \tau \in \{1, 2, ..., L-1\} \tag{27}$$

with boundary conditions

$$Q_{ij}^1 = K_{ij} \tag{28}$$

where $K_{ij} \in \Re_+^1$ is the fixed travel demand for the OD pair $(i,j) \in \mathcal{W}$ for the first day and χ_{ij} is the so-called fitness level. The operator $[x]^+$ is shorthand for $\max[0, x]$. The parameter s_{ij}^{τ} is related to the rate of change of inter-day travel demand.

3.1 Within-Day Fixed Point Formulation

We have already commented that an algorithm for the within-day differential variational inequality is needed if the dual time scale model is to be solved.

Solution of the within-day differential variational inequality, as we have also mentioned, is complicated by the fact that the effective delay operator

$$\Psi(t,h) = (\Psi_p(t,h) : p \in \mathcal{W})$$

is typically neither monotonic nor differentiable. Consequently, we must select an algorithm that places minimal restrictions on $\Psi(t,h)$. One such category of algorithms is that of iterative methods in Hilbert space for a fixed point equivalent of the within-day differential variational inequality

$$\left.\begin{array}{c} \text{find } h^{\tau*} \in \Lambda_\tau(Q^\tau) \text{ such that} \\ \\ \sum_{p\in\mathcal{P}} \displaystyle\int_{(\tau-1)\cdot\Delta}^{\tau\cdot\Delta} \Psi_p(t,h^{\tau*}) \left(h_p^\tau - h_p^{\tau*}\right) dt \geq 0 \\ \\ \forall h^\tau \in \Lambda_\tau(Q^\tau) \end{array}\right\} DVI(\Psi, \Lambda_\tau, \Delta) \tag{29}$$

for every $\tau \in \Upsilon$. We will use the notation $DUE(\Psi, \Lambda_\tau, \Delta)$ for the within-day dynamic user equilibrium equivalent to $DVI(\Psi, \Lambda_\tau, \Delta)$ defined in (29) above. With the preceding background, we are in a position to state and prove a result that permits the solution of the differential variational inequality (29) to be obtained by solving an appropriate fixed point problem. That result is:

Theorem 2 *Fixed point re-statement. Assume, for each $\tau \in \Upsilon$, that $\Psi_p(\cdot, h^\tau)$: $[(\tau-1)\cdot\Delta, \tau\cdot\Delta] \longrightarrow \Re^1_+$ is measurable for all $p \in \mathcal{P}$, $h^\tau \in \Lambda_\tau(Q^\tau)$. Then, for each $\tau \in \Upsilon$, the fixed point problem*

$$h^\tau = P_{\Lambda_\tau(Q^\tau)}\left[h^\tau - \alpha\Psi(t,h^\tau)\right], \tag{30}$$

is equivalent to $DVI(\Psi, \Lambda_\tau, \Delta)$ where $P_{\Lambda_\tau(Q^\tau)}[\cdot]$ is the minimum norm projection onto $\Lambda_\tau(Q^\tau)$ and $\alpha \in \Re^1_{++}$.

Proof: Friesz et al. (2008) give a formal proof of this result.

3.2 The Within-Day Algorithm

Naturally Theorem 2 suggests the following algorithm:

Fixed Point Algorithm for $DUE(\Psi, \Lambda_\tau, \Delta)$

Step 0. Initialization. Select $h^{\tau,0}$ and the rule for generating the sequence $\{\beta_k\}$. Also select a stopping tolerance $\epsilon \in \Re^1_{++}$. Set $k = 0$.

Step 1. Major iteration. Compute

$$h^{\tau,k+1} = P_{\Lambda_\tau(Q^\tau)}\left[h^{\tau,k} - \alpha\Psi\left(t, h^{\tau,k}\right)\right]$$

Step 2. Stopping test. If

$$\left\| h^{\tau,k+1} - h^{\tau,k} \right\| \leq \epsilon$$

stop and declare

$$h^{\tau,*} \approx h^{\tau,k+1}$$

Otherwise set $k = k + 1$ and go to Step 1.

3.3 The Dual Time Scale Algorithm

It is appropriate for us to now provide a summary of the time stepping method intrinsic to the dual time scale model and its relationship to the within-day fixed point algorithm. The main objective of the time stepping method is to separate the day-to-day dynamics from a sequence of within-day DUE problems, so that exactly one DUE problem is faced for each day. Recall the day-to-day demand growth model of our interest is

$$Q_{ij}^{\tau+1} = \mathcal{F}_{ij}\left(Q_{ij}^{\tau}, h^1, h^2, ..., h^{\tau}; \theta\right) \quad \forall (i,j) \in \mathcal{W},\ \tau \in \{1, 2, ..., N-1\} \tag{31}$$

$$Q_{ij}^{\tau} \geq 0 \quad \forall (i,j) \in \mathcal{W},\ \tau \in \{2, ..., N\} \tag{32}$$

$$Q_{ij}^{1} = K_{ij} \quad \forall (i,j) \in \mathcal{W} \tag{33}$$

The algorithm itself has the form given below:

Complete Algorithm for the Dual Time Scale Model

Step 0. Initialization. Given $Q_{ij}^{1,*} = K_{ij}$ for all $(i,j) \in \mathcal{W}$, choose the vector of model parameters θ. Set $\tau = 1$.

Step 1. Solving the Within-Day Model. Solve $DUE(\Psi, \Lambda_{\tau}(Q^{\tau *}), \Delta)$ for day τ by the fixed point algorithm in Section 3.2. Call the solution $h^{\tau,*}$.

Step 2. Update Demand. With the equilibrium information in hand, compute the travel demand for the next day according to

$$Q_{ij}^{\tau+1,*} = \mathcal{F}_{ij}\left(Q_{ij}^{\tau,*}, h^{1,*}, h^{2,*}, ..., h^{\tau,*}; \theta\right) \quad \forall (i,j) \in \mathcal{W}$$

Step 3. Time Stepping. If $\tau = N$, stop. Otherwise set $\tau = \tau + 1$ and go to Step 1.

4 Dynamic Network Loading

The problem of finding link activity when travel demand and departure rates (path flows) are known is commonly referred to as the dynamic network loading problem. Effective path delays are constructed from arc delays that, directly or indirectly, depend on arc activity; moreover, activity on a given arc is influenced by the delays on paths that utilize that arc. Thus, dynamic network loading is intertwined with the determination of path delays. Recall that, in our formulation and computational scheme for the dual time scale model presented above, all we require of the effective path delay operators is that they are measurable and nonnegative. As such our formulation of dual time scale dynamic user equilibrium can accommodate effective path delays derived from virtually any dynamic network loading procedure and any model of queueing that imputes arc delays.

For our calculations, we have employed two network loading models based on the following:

1. the point queue model presented by Friesz et al. (1993) and Friesz and Mookherjee (2006); and

2. the cell transmission model as implemented by Lo and Szeto (2002).

We employ these models to compute path delays for any given vector of departure rates h according to procedures described in Lo and Szeto (2002) and Friesz et al. (2008).

5 Numerical Examples

In this section, we present the results of fixed point algorithm when applied to the well known 76 arc Sioux Falls network (see Figure 1) for the two network loading algorithms mentioned in the preceding section. Our example has seven origin-destination pairs. The key exogenous data are given in Table 1 and Table 2. In detail, Table 1 provides the arc costs and capacities for the Sioux Falls network. In Table 2, the number of paths and desired arrival time is provided for each OD pair. The desired arrival time is based on the continuous time commuting period from 8:00 to 9:20. The computed exit flow rates for each arc of two representative paths are presented in Figure 2 and Figure 3. From the same figure (and other plots not included here for the sake of brevity) we see that our numerical solutions are *bona fide* dynamic user equilibria. In Figure 4 we show how demand evolves on a day-to-day basis.

5.1 Performance of the Fixed Point Algorithm

The numerical example was solved by the continuous time DUE fixed point algorithms employing two network loading models. The two network loading models used were the point queue model and the cell transmission model. The solution

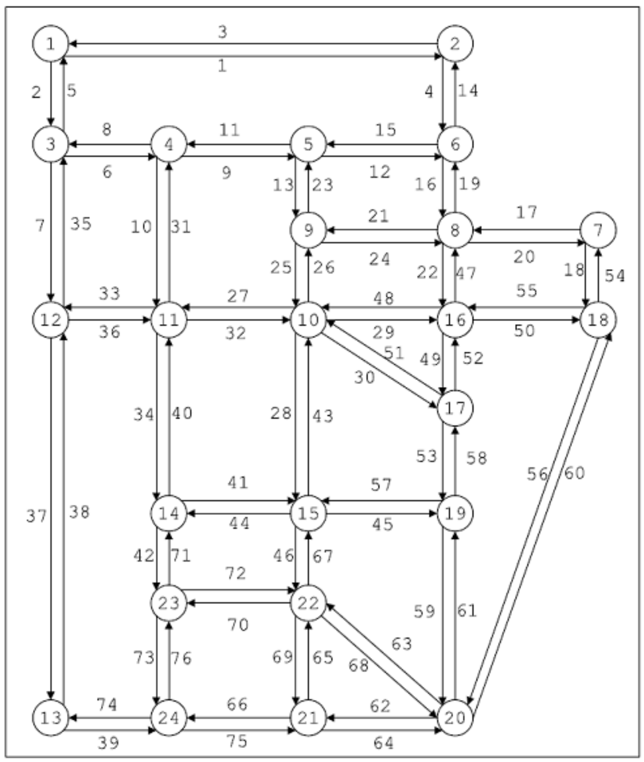

Figure 1: Sioux Falls Network

Arc	Cost	Capacity	Arc	Cost	Capacity	Arc	Cost	Capacity
1	6	90	27	3	90	52	2	90
2	2	90	28	4	90	53	6	90
3	6	90	29	2	90	54	4	90
4	2	90	30	6	90	55	2	90
5	2	90	31	3	90	56	16	90
6	2	90	32	3	90	57	2	90
7	4	90	33	4	90	58	6	90
8	2	90	34	3	90	59	4	90
9	2	90	35	4	90	60	16	90
10	3	90	36	4	90	61	4	90
11	2	90	37	14	90	62	4	90
12	10	90	38	14	90	63	6	90
13	2	90	39	4	90	64	4	90
14	2	90	40	3	90	65	8	90
15	10	90	41	4	90	66	4	90
16	2	90	42	2	90	67	2	90
17	2	90	43	4	90	68	6	90
18	4	90	44	4	90	69	8	90
19	2	90	45	2	90	70	2	90
20	2	90	46	2	90	71	2	90
21	4	90	47	2	90	72	2	90
22	2	90	48	2	90	73	2	90
23	2	90	49	2	90	74	4	90
24	4	90	50	2	90	75	4	90
25	2	90	51	6	90	76	2	90
26	2	90						

Table 1: Exogenous Arc Data for Sioux Falls Network

OD pairs	Number of Paths	Desired Arrival Time
$(1, 20)$	55	8:45
$(13, 20)$	4	8:50
$(3, 15)$	6	9:00
$(12, 18)$	1	9:05
$(2, 13)$	15	9:10
$(9, 19)$	5	9:15
$(4, 7)$	6	9:20

Table 2: Exogenous Data for Sioux Falls Network

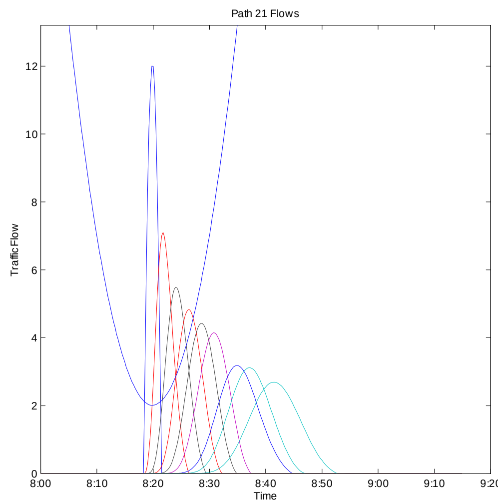

Figure 2: Traffic Flow and Travel Cost for Path 21 for the Point Queue Model

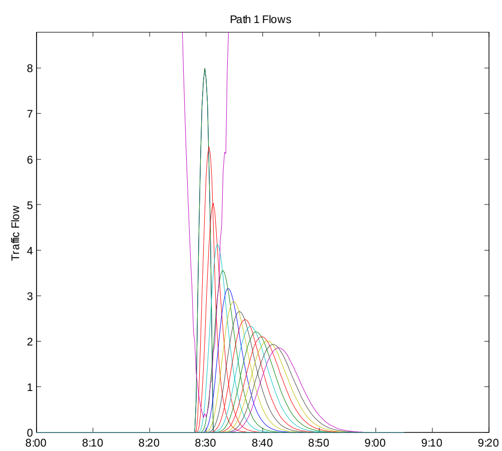

Figure 3: Traffic Flow and Travel Cost for Path 1 for the Cell Transmission Model

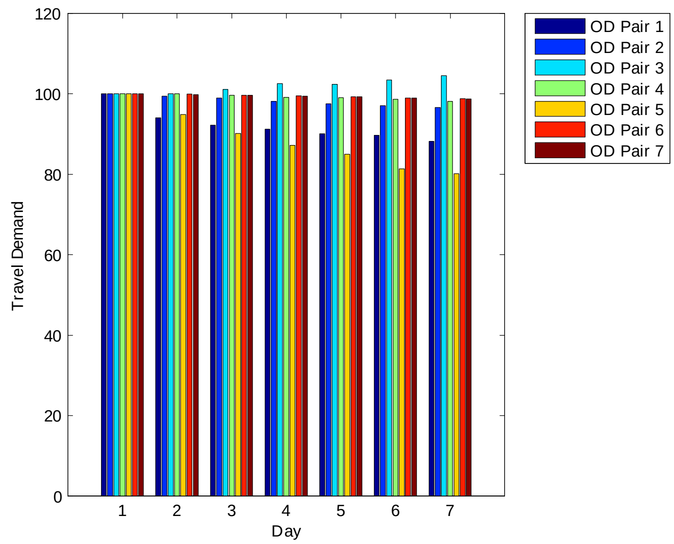

Figure 4: Travel Demand Changes for 7 OD Pairs

has been coded in MATLAB 7 and GAMS, and solved on a standard desktop computer with the following attributes: Windows Vista with Intel Core2 Duo 2.20GHz and 1.5GB RAM. The day-to-day time stepping used to update demand is extremely fast and therefore does not significantly affect computation time. In addition, each within-day dynamic user equilibrium calculations and successive within-day calculation in day-to-day evolution dynamics becomes faster since a warm-start protocol is employed whereby a trial solution based on the previous within-day flow pattern is used. To measure the performance of the fixed point algorithm for the point queue model and the cell transmission model, we collected the number of iterations for each day. Table 3 provides the number of such iterations for the fixed point algorithm for each model.

As shown in Figure 5, the algorithm significantly decreases the error in the first several iterations.

6 Concluding Remarks

We have presented a dual time scale model of dynamic network traffic flows that integrates a day-to-day demand growth model with a differential variational inequality formulation of within-day dynamic user equilibrium; this model is compatible with a variety of network loading models. The differential variational inequality formulation we have given for within-day dynamic user equilibrium offers advantages: (1) the differential variational inequality may be very easily

	Point Queue Model	Cell Transmission Model
Day 1	15 times	18 times
Day 2	15 times	16 times
Day 3	15 times	17 times
Day 4	15 times	17 times
Day 5	14 times	17 times
Day 6	13 times	16 times
Day 7	13 times	16 times
Total Iteration	87 times	117 times

Table 3: Algorithm performance on example problems

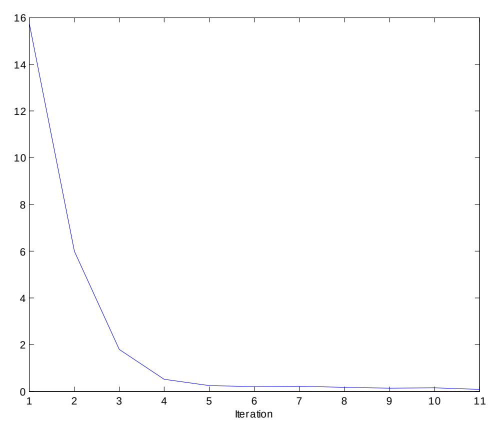

Figure 5: Convergence plot for the Fixed Point algorithm

analyzed the minimum principle from optimal control theory; (2) our fixed point algorithm is able to consider non-differentiable and non-analytic path delay operators; and (3) the rapidly growing literature on differential variational inequalities (see Pang and Stewart, 2008, for a review) will likely yield additional computational tools in the years ahead that may be exploited to find DUE flows.

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