PATH-BASED APPROACHES TO ROBUST NETWORK DESIGN PROBLEMS CONSIDERING BOUNDEDLY RATIONAL NETWORK USERS

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ABSTRACT
Network users may choose non-shortest paths, when (1) they satisfice with sub-optimal routes, or (2) they have perception errors of the decision environment. The notion of generalized bounded rationality has been recently proposed to create a unified framework for these two sources of behavioral uncertainty in route choices. When the notion of generalized bounded rationality is used in robust network design problems, we obtain a bi-level optimization problem with the min-max objective function at the upper level, with three layers of optimization in total. In this paper, we derive equivalent single-level path-based formulations that are readily solvable by available optimization libraries. We show how to incorporate them into robust multi-commodity network design problems in hazardous materials transportation.

Keywords: networks and graphs; satisficing; network design; hazardous materials; bi-level optimization

INTRODUCTION

The route choice behavior of a driver is a fundamental and critical component of the network design problem. With growing evidence that drivers do not necessarily utilize the shortest or minimum cost path [1, 2], route choice models have been extensively studied and the incorporation of route choice behaviors into network design problems is attracting more attention [3, 4].

Bounded rationality is a major route choice model among the various ones employed. It can be traced back to the research by Simon [5] to "make fairly drastic revision of the rational man" notion assumed in economics. The paper recognizes the existence of satisfaction with certain aspiration levels and the perceived subjective environment compared to the objective environment [6] due to the complexity of information gathering and preferences to payoffs.

Mahmassani and Chang [7] first introduced bounded rationality (BR) to transportation research. BR assumes that drivers act as if there is an *indifference band*, resulting in multiple user equilibria (UE). Considering this property, Guo and Liu [8] study the effect of opening a bridge using bounded rationality user equilibrium (BRUE). Di et al. [9] study how BRUE affects the Braess paradox. Furthermore, Di et al. [10] theoretically explore the properties of BRUE under the static traffic assignment problem by dividing BRUE into multiple subsets. In the dynamic setting, Wu et al. [11] consider BR in an urban railway network assuming that travelers update their path only if the difference is beyond a certain value on a daily basis. For a dynamic traffic assignment problem considering route choice and departure time together, Szeto and Lo [12] provide a discrete time nonlinear complementarity formulation. Han et al. [13] give a variational inequality formulation and study the existence of solution, as well as providing three algorithms to solve the problem.

The notion of BR is also incorporated in the congestion toll pricing problem. Due to the non-convexity and non-uniqueness of BRUE, robust approaches are used by considering the risk preferences of the decision makers, leading to risk averse and risk prone toll pricing models [14, 15]. Moreover, Di et al. [15] examine the topology of the BRUE set in robust toll pricing problems and established the property in terms of existence of the solution. Alternatively, Guo [16] shows how to use a sequence of toll regulations to achieve a unique user equilibrium when BR bounds are homogeneous for diverse origin destination (OD) pairs and how to reduce the BRUE set in the heterogeneous case.

The boundedness of rationality comes from two sources: (1) satisficing behavior and (2) subjective perception error of the objective environment [17]. Most transportation research, however, is based on the first source of bounded rationality; that is, a driver is satisfied with a route as long as the route's utility reaches his aspiration level. Sun et al. [17] address bounded rationality from the second source by proposing a perception error model where a driver perceives the link cost with error and the error belongs to a closed and bounded set. By using certain special perception error sets, they show that optimizing with perfect information is equivalent to satisficing with perception error. This allows the study of the two bounded rationality sources to be unified into a single framework and motivates the definition of generalized bounded rationality (GBR). Sun et al. [17] then show the effectiveness of GBR in robust multi-commodity network design problems using a link-based model. Sun et al. [17] proposed a modified iterative cutting plane algorithm based on the work of Gzara [18].

Contributions

Our main contributions are the single-level linear optimization reformulations that are equivalent to the link-based formulation of Sun et al. [17] for robust network design problems incorporating GBR. The link-based formulation is bi-level in the sense that there are two distinct groups of decision makers: a network operator (upper-level) and drivers (lower-level). The upper-level objective has a min-max structure to make decisions robust against behavioral uncertainty of drivers. The lower-level problem is in the form of the shortest-path problem. The resulting problem is hence a bi-level min-max problem with three layers of optimization in total. A key challenge in the single-level reformulations is to capture drivers' uncertain behavior described in the lower-level problem, while considering the min-max robust objective function in the upper-level problem. We propose novel reformulation techniques to tackle this challenge.

While link-based formulations require specialized algorithms such as cutting plane, our single-level path-based reformulations are readily solvable by standard optimization solvers such as CPLEX and Gurobi. Our path-based reformulations are more effective in small networks and in situations that particular path sets should be chosen for the study of network design problems. We illustrate the path-based formulations in the context of the hazmat (hazardous materials) network design problem (HNDP) proposed by Kara and Verter [19], in which the upper level model (government) decides which road links to close so as to minimize risk considering that the lower level model (hazmat carriers) will choose the minimum cost routes. Verter and Kara [20] provide a path-based model for HNDP to allow compromise between government and the hazmat carriers, which gives insights on the path-based formulations of GBR in the robust network design problem.

Structure of the Paper

The remainder of the paper is organized as follows. In Section 3, we give more details on how satisficing behaviors and perception error could be unified into a general framework: GBR. In Section 4, the general structure of robust network design problems is discussed. In Section 5, we provide path formulations for robust network design problems considering GBR. Finally, we give numerical results in Section 6 and a conclusion in Section 7.

SATISFICING AND PERCEPTION ERROR

In this section, we give more details on the two sources of boundedness: satisficing and perception error; and the notion of generalized bounded rationality by Sun et al. [17].

Compared to the literature on bounded rationality in transportation research, Sun et al. [17] study route choice behaviors by focusing on the second source of boundedness. For a given network $\mathcal{G}(\mathcal{N}, \mathcal{A})$, let $i \in \mathcal{N}$ be the set of nodes and $(i, j) \in \mathcal{A}$ be the set of links. The cost of each link (i, j) is c_{ij}. Let x_{ij} denote whether link (i, j) is chosen by the driver or not. Then for a single origin destination (OD) pair o and d, we have

$$\mathbb{X} = \left\{ x : \sum_{(i,j)\in\mathcal{A}} x_{ij} - \sum_{(j,i)\in\mathcal{A}} x_{ji} = b_i \ \forall i \in \mathcal{N}, \quad x_{ij} \in \{0, 1\} \ \forall (i, j) \in \mathcal{A} \right\} \tag{1}$$

where $b_i = 1$ for $i = o$, $b_i = -1$ for $i = d$, and $b_i = 0$ for all other nodes $i \in \mathcal{N}$.

Given the above definition, Sun et al. [17] introduce the following satisficing route choice behaviors: Additive Satisficing (A-Sat), Multiplicative Satisficing (M-Sat) and Subpath Multiplicative Satisficing (SM-Sat).

Definition 1 (Sun et al. 17)**.** A path is called an *additive satisficing (A-Sat) path*, if the path can be represented by a vector $x \in \mathbb{X}$ such that

$$\text{(A-Sat)} \qquad \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} \leq c^0 + E \tag{2}$$

where c^0 is the minimum cost path utility value and E is a nonnegative constant for the additive indifference band.

Definition 2 (Sun et al. 17)**.** A path is called a *multiplicative satisficing (M-Sat) path*, if the path can be represented by a vector $x \in \mathbb{X}$ such that

$$\text{(M-Sat)} \qquad \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} \leq (1+\kappa)c^0 \tag{3}$$

where $\kappa \in [0, 1)$ is a constant for the multiplicative indifference band.

Definition 3 (Sun et al. 17)**.** A path is called a *subpath multiplicative satisficing (SM-Sat) path* with a constant κ, if every subpath of the path is an M-Sat path with the same constant κ between the corresponding origin and destination nodes.

Then a perception error (PE) model is proposed by Sun et al. [17] as the following:

$$\text{(PE)} \qquad \min_{x\in\mathbb{X}} \sum_{(i,j)\in\mathcal{A}} (c_{ij} - \varepsilon_{ij})x_{ij} \tag{4}$$

for some constant cost vector $\varepsilon \in \mathcal{E}$. The vector ε denotes the network user's *perception error* of link cost, and the set $\mathcal{E}$ is the *set of uncertain perception errors*. From the formulation of PE, we notice that the model is optimizing, however there is error in the subjective perception.

By comparing the satisficing route choice behaviors and the PE model, for some special cases of $\mathcal{E}$, Sun et al. [17] prove that satisfying under perfect information is equivalent to optimizing with perception error. Particularly,

$$\text{PE} + \mathcal{E}_A \Longleftrightarrow \text{A-Sat}, \qquad \mathcal{E}_A = \left\{\varepsilon : \sum_{(i,j)\in\mathcal{A}} \varepsilon_{ij} \leq E, \varepsilon_{ij} \geq 0\right\}$$

$$\text{PE} + \mathcal{E}_M \Longleftrightarrow \text{M-Sat}, \qquad \mathcal{E}_M = \left\{\varepsilon : \sum_{(i,j)\in\mathcal{A}} \varepsilon_{ij} \leq \kappa c^0, \varepsilon_{ij} \geq 0\right\}$$

$$\text{PE} + \mathcal{E}_L \Longleftrightarrow \text{SM-Sat}, \qquad \mathcal{E}_L = \left\{\varepsilon : 0 \leq \varepsilon_{ij} \leq \frac{\kappa}{1+\kappa}c_{ij}, \forall (i,j) \in \mathcal{A}\right\}.$$

Based on the above, a more general framework of generalized bounded rationality (GBR) is proposed when the perception error could be modeled by any closed and bounded set. For example, some general sets could be:

$$\text{Hybrid Error Set, } \mathcal{E}_H = \left\{\varepsilon : \sum_{(i,j)\in\mathcal{A}} \varepsilon_{ij} \leq E, \sum_{(i,j)\in\mathcal{A}} \varepsilon_{ij} \leq \kappa c^0, \varepsilon_{ij} \geq 0\right\}, \tag{5}$$

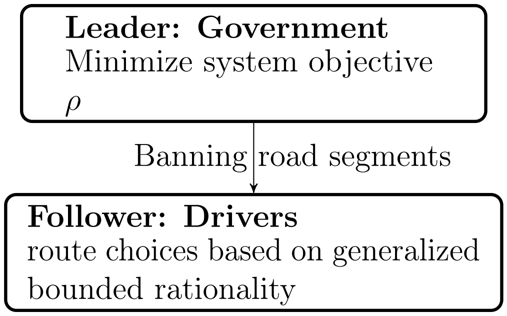


FIGURE 1: Robust Network Design Problem Structure

$$\text{Box Error Set, } \mathcal{E}_B = \left\{ \varepsilon : l_{ij} \leq \varepsilon_{ij} \leq u_{ij} \quad \forall (i,j) \in \mathcal{A} \right\}, \tag{6}$$

$$\text{Ellipsoidal Error Set, } \mathcal{E}_E = \left\{ \varepsilon : ||Q^{-1/2}\varepsilon||_2 \leq \xi \right\}. \tag{7}$$

After the definition of GBR, Sun et al. [17] model a link-based robust multi-commodity network design problem to illustrate its effectiveness and utilize a cutting plane algorithm to solve the problem. In the next section, we introduce the robust multi-commodity network design problem.

ROBUST NETWORK DESIGN PROBLEM

The network design problem in transportation has a structure of achieving certain objectives by anticipating and interfering the behavior of drivers, mostly route choice behaviors. Due to this nature, a leader-follower bilevel model can be formulated to represent the problem. The leader makes decisions to obtain its best objective by considering the follower's route choices. Much of the literature in the network design problem assumes drivers utilize the minimum cost routes (either measured in distance, time etc. or a utility function considering several factors). However, it is nearly impossible to measure the drivers' route choices exactly. Due to the uncertainty in routes chosen by the drivers, a robust approach could be applied to achieve a best worst case result. Particularly, we consider network design problems in which the leader could close certain road segments, considering boundedly rational choice of drivers. The structure of the robust network design problem is shown in Figure 1.

We introduce the bi-level min-max link-based robust network design formulation of Sun et al. [17]:

$$\min_{y} \left(\alpha \sum_{(i,j)\in\mathcal{A}} (1 - y_{ij}) + \max_{\varepsilon} \sum_{(i,j)\in\mathcal{A}} \sum_{s\in\mathcal{S}} n^s \rho_{ij}^s x_{ij}^s \right) \tag{8}$$

$$y_{ij} \in \{0,1\} \quad \forall (i,j) \in \mathcal{A} \tag{9}$$

$$\varepsilon^s \in \mathcal{E}^s \quad \forall s \in \mathcal{S} \tag{10}$$

$$\min_{x} \sum_{(i,j)\in\mathcal{A}} (c_{ij} - \varepsilon_{ij}^s) x_{ij}^s \tag{11}$$

$$\text{s.t.} \sum_{(i,j)\in\mathcal{A}} x_{ij}^s - \sum_{(j,i)\in\mathcal{A}} x_{ji}^s = b_i^s \quad \forall i \in \mathcal{N} \tag{12}$$

$$x_{ij}^s \leq y_{ij} \quad \forall (i,j) \in \mathcal{A}, s \in \mathcal{S} \tag{13}$$

$$x_{ij}^s \in \{0,1\} \quad \forall (i,j) \in \mathcal{A}, s \in \mathcal{S} \tag{14}$$

This problem is a bi-level optimization wherein the upper-level problem has a min-max robust objective function. Three are three layers of optimization in total. In the upper-level problem, the decision variable is binary y_{ij} that denotes whether link (i,j) is open when $y_{ij} = 1$ or closed otherwise. For each shipment $s \in \mathcal{S}$, the transport demand is n^s and its route-choice is made by each carrier for the shipment independently. Since the upper-level decision maker is uncertain about the suboptimal boundedly-rational route-choice of drivers, the robust objective function in the min-max form is considered. The link property ρ_{ij}^s is the measure of per-unit disutility to the upper-level decision maker from shipment s being traveling on link (i,j). In the upper-level objective function, the first term is added with positive constant α to minimize the number of links to be closed. The lower-level problem (11)–(14) models the shortest-taking behaviors of drivers under the perception error ε_{ijs} and closed links y_{ij}. In constraint (10), $\mathcal{E}^s$ is the chosen perception error set, for which we consider $\mathcal{E}_H$, $\mathcal{E}_B$, and $\mathcal{E}_E$.

PATH-BASED FORMULATIONS FOR ROBUST NETWORK DESIGN PROBLEMS

In this section, we provide path-based reformulations that are equivalent to the link-based bi-level network design problems (8)–(14) for certain perception error sets. In particular, we consider $\mathcal{E}_H$ and $\mathcal{E}_B$. Note that $\mathcal{E}_H$ includes $\mathcal{E}_A$ and $\mathcal{E}_M$, and $\mathcal{E}_B$ includes $\mathcal{E}_L$. We present the path-based reformulations as mixed-integer linear programming (MILP) problems that can be solved by popular optimization solvers such as CPLEX and Gurobi. These path-based reformulations will be especially useful when the number of available paths for each OD pair is relatively small and when a certain set of paths needs to be studied. For example, the decision makers want to define the path set selectively, which would be impossible for link-based models.

In order to enumerate the paths, we adopt a loopless K shortest paths algorithm. Particularly, we use the improved algorithm of [21] by Martins and Pascoal [22], which has the computational complexity of $\mathcal{O}\left[K|\mathcal{N}|(|\mathcal{A}| + |\mathcal{N}| \log |\mathcal{N}|)\right]$ and works in both directed and undirected graphs. By changing the stopping rule to exceeding a certain cost threshold instead of the number of paths reaching K, we have the flexibility of generating the set of paths.

Path-Based Formulation with $\mathcal{E}_H$

Before we present the path-based reformulation of the robust network design problem considering the hybrid perception error set $\mathcal{E}_H$ (including $\mathcal{E}_A$ and $\mathcal{E}_M$), we first explore how to obtain the general bounded rationality path set based on $\mathcal{E}_H$.

By the definition of the hybrid error set, we generate paths whose lengths are less than $\min\{c^0 + E, (1+\kappa)c^0\}$. We can utilize the loopless K shortest paths algorithm with a sufficiently larger number K. We rank the paths generated by the K loopless shortest path algorithm by their length in ascending order. Let $\mathcal{P}^s$ be the path set with K loopless shortest paths for shipment $s \in \mathcal{S}$ and let p_k^s be the k-th path in $\mathcal{P}^s$ with N_k^s number of links. Let $\mathcal{K} = \{1, 2, \cdots, K\}$ be the index set of the generated paths.

A key challenge is that the length of the *available* shortest path, c^0, changes as the network design changes and some paths become unavailable. To handle this varying nature of the problem, we construct a set of paths

$$\Psi_k^s = \left\{ p \in \mathcal{P}^s : (\text{length of } p) \leq \min\{c_k^s + E, (1+\kappa)c_k^s\} \right\}$$

for each $k \in \mathcal{K}$, $s \in \mathcal{S}$, where c_k^s is the length of path p_k^s.

Now we propose a path-based formulation of the robust network design problem. For each path p_k^s, besides its cost c_k^s, another important path characteristic is ρ_k^s, any measure that the network designer wants to minimize (see Figure 1). The structure of network design problem could be summarized as utilizing certain regulation policy (in our case, closing certain road links) to achieve paths with minimum objective $\sum_{s \in \mathcal{S}} \rho_k^s$ while drivers will take the routes belonging to the generalized bounded rationality path sets.

We represent each path p_k^s by a binary flow vector $x^{ks} \in \mathbb{X}$. Accordingly, we define the arc set for path p_k^s as $\mathcal{A}_k^s = \{(i,j) \in \mathcal{A} : x_{ij}^{ks} = 1\}$. In order to formulate the path-based models, we define the following decision variables:

y_{ij}: binary variable indicating whether link $(i,j) \in \mathcal{A}$ is open for shipments,

X_k^s: binary variable indicating whether path p_k^s is used for shipment s,

Z_k^s: binary variable indicating whether path p_k^s is available for shipment s.

Then we can formulate the path-based robust network design problem with $\mathcal{E}_H$ as follows:

$$\min_{X,y,Z} \left(\alpha \sum_{(i,j)\in\mathcal{A}} (1 - y_{ij}) + \max_{m \in \Psi_k^s} \sum_{s\in\mathcal{S}} \sum_{k\in\mathcal{K}} n^s \rho_m^s X_k^s \right), \tag{15}$$

$$\text{s.t.} \sum_{k\in\mathcal{K}} X_k^s = 1, \quad \forall s \in \mathcal{S}, \tag{16}$$

$$X_k^s \leq Z_k^s, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, \tag{17}$$

$$Z_k^s \leq y_{ij}, \quad \forall s \in \mathcal{S}, k \in \mathcal{K} \cup \Psi_k^s, (i,j) \in \mathcal{A}_k^s, \tag{18}$$

$$Z_k^s \geq \sum_{(i,j)\in\mathcal{A}_k^s} y_{ij} - N_k^s + 1, \quad \forall s \in \mathcal{S}, k \in \mathcal{K} \cup \Psi_k^s, \tag{19}$$

$$X_k^s \geq Z_k^s - \sum_{n=1}^{k-1} Z_n^s, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, \tag{20}$$

$$X_k^s, Z_k^s \in \{0,1\}, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, \tag{21}$$

$$y_{ij} \in \{0,1\}, \quad \forall (i,j) \in \mathcal{A}. \tag{22}$$

The objective (15) of the model is to minimize the worst objective value desired by the leader among the general bounded rational path set Ψ_k^s while minimizing the number of closed arcs. Constraints (16) guarantee only one path is chosen for each OD pair. Constraints (17) allow carriers to choose a path only if it is available. Constraints (18) forbid carriers from taking the path if any of its arcs are closed while constraints (19) allow a path to be available if all of its arcs are open. Constraints (20) enforce that the path with the smallest index should be chosen. Constraints (21)–(22) ensure the decision variables to be binary.

The model (15)–(22) is a bilevel program with a min-max structure. We reformulate it as an MILP model. Let

$$\tilde{\rho}_k^s = \begin{cases} \max\left\{\rho_m^s : p_m^s \in \Psi_k^s, Z_m^s = 1\right\}, & X_k^s = 1, \\ 0, & X_k^s = 0. \end{cases} \tag{23}$$

Then the min max structure of objective (15) can be replaced by

$$\min_{X,y,Z,\tilde{\rho}} \sum_{s\in\mathcal{S}} \sum_{k\in\mathcal{K}} n^s \tilde{\rho}_k^s.$$

In order to formulate as an MILP, equation (23) can be modeled using the following constraints:

$$X_k^s Z_m^s \rho_m^s \leq \tilde{\rho}_k^s, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, m \in \Psi_k^s. \tag{24}$$

Note constraints (24) are nonlinear. Letting $W_{mk}^s = X_k^s Z_m^s$ and $W_{mk}^s \in \{0,1\}$, we achieve a single level MILP as follows:

$$\min_{X,y,Z,\tilde{\rho}} \sum_{s\in\mathcal{S}} \sum_{k\in\mathcal{K}} n^s \tilde{\rho}_k^s + \alpha \sum_{(i,j)\in\mathcal{A}} (1 - y_{ij}), \tag{25}$$

$$(16) - (22),$$

$$W_{mk}^s \rho_m^s \leq \rho_k^s, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, m \in \Psi_k^s, \tag{24a}$$

$$- X_k^s + W_{mk}^s \leq 0, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, m \in \Psi_k^s, \tag{24b}$$

$$- Z_m^s + W_{mk} \leq 0, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, m \in \Psi_k^s, \tag{24c}$$

$$- W_{mk} + Z_m^s + X_k^s \leq 1, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, m \in \Psi_k^s, \tag{24d}$$

where (24a)–(24d) are the linearized constraints for (24).

Path-Based Formulation with $\mathcal{E}_B$

For any path in the generalized bounded rationality path set based on $\mathcal{E}_B$ (including $\mathcal{E}_L$), it needs to be a shortest path for at least one realization of $\mathcal{E}_B$. We note that each element ε_{ij} is constrained in $\mathcal{E}_L$, while the sum of ε_{ij} is constrained in $\mathcal{E}_H$. To handle this difference, we give a necessary and sufficient condition for obtaining this path set based on Lemma 2 of Sun et al. [17] and Proposition 2.3 in Karaşan et al. [23].

Proposition 1. *A path $\bar{p}$ with flow vector $\bar{x} \in \mathbb{X}$ is a solution to the perception-error model* (4) *for some $\varepsilon \in \mathcal{E}_B$ if and only if it can be perceived as the shortest path when the lengths of all arcs $(i,j) \in \overline{\mathcal{A}} = \{(i,j) \in \mathcal{A} : \bar{x}_{ij} = 1\}$ are at their lower bounds $c_{ij} - u_{ij}$ and the lengths of all the remaining arcs are at their upper bounds $c_{ij} - l_{ij}$.*

Now based on Proposition 1, for path $\bar{p}$ with flow vector $\bar{x}$, by setting the lengths of all arcs $(i,j) \in \overline{\mathcal{A}}$ at their lower bounds $c_{ij} - u_{ij}$ and the lengths of all the remaining arcs at their upper bounds $c_{ij} - l_{ij}$, we can compare path $\bar{p}$ with all paths enumerated using the K loopless shortest path algorithm to test if it is a solution to perception-error model (4) for some $\varepsilon \in \mathcal{E}_B$. Then by doing this procedure for all the paths, we can obtain the general bounded rationality path set with box error $\mathcal{E}_B$.

In order to formulate the robust network design problem with box perception error $\mathcal{E}_B$, we first rank the paths $\mathcal{P}^s$ we generated for each OD pair s in descending order of the path characteristic ρ_k^s which is of interest by the upper level objective. Let $\mathcal{A}_k^s$ be the set of arcs belonging to path p_k^s and $\mathcal{A}_{k'}^s$ be the set of arcs belonging to path $p_{k'}^s$. Based on Proposition 1, for path $p_k^s \in \mathcal{P}^s$ we define

$$L_k^s = \sum_{(i,j)\in\mathcal{A}_k^s} (c_{ij}^s - u_{ij}^s),$$
$$L_{kk'}^s = \sum_{(i,j)\in\mathcal{A}_k^s\cap\mathcal{A}_{k'}^s} (c_{ij}^s - u_{ij}^s) + \sum_{(i,j)\in\mathcal{A}_{k'}^s\setminus\mathcal{A}_k^s} (c_{ij}^s - l_{ij}^s) \quad \forall p_{k'}^s \in \mathcal{P}^s.$$

L_k^s is the length of path p_k^s when all its arcs are at their lower bounds and $L_{kk'}^s$ is the length of path $p_{k'}^s$ when all its arcs are at their upper bounds except the common arcs with path p_k^s. We then design an auxiliary matrix Φ^s for each shipment $s \in \mathcal{S}$ to identify paths that can be perceived as shortest:

$$\Phi^s = \begin{bmatrix} \phi_{11}^s & \phi_{12}^s & \cdots & \phi_{1K}^s \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{k1}^s & \phi_{k2}^s & \cdots & \phi_{kK}^s \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1}^s & \phi_{n2}^s & \cdots & \phi_{KK}^s \end{bmatrix}, \text{ where } \phi_{kk'}^s = \begin{cases} 0, & L_k^s > L_{kk'}^s, \\ 1, & L_k^s \le L_{kk'}^s, \end{cases} \qquad \forall k, k' = 1, ..., K.$$

Let V_k^s be a binary variable indicating whether path p_k^s is available and included in the general bounded rationality path set with $\mathcal{E}_B$ for shipment s or not. We then introduce the following proposition.

Proposition 2. *Suppose that $Z_k^s = 1$ if path p_k^s is available and $Z_k^s = 0$ otherwise. For any path p_k^s, the definition of V_k^s is equivalent to the following constraints:*

$$V_k^s \le \phi_{kk'}^s + (1 - Z_{k'}^s), \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, k' \in \mathcal{K}, \tag{26}$$
$$V_k^s \ge \sum_{k'\in\mathcal{K}} [(\phi_{kk'}^s - 1) Z_{k'}^s] + Z_k^s, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, \tag{27}$$
$$V_k^s \le Z_k^s. \quad \forall s \in \mathcal{S}, k \in \mathcal{K}. \tag{28}$$

Proof. For constraints (27), we have:

1. $V_k^s \ge 1$ if the path is available ($Z_k^s = 1$) and $\phi_{kk'}^s = 1$ for all the available paths that $Z_{k'}^s = 1$. Since V_k^s is binary, $V_K^s = 1$.
2. $V_k^s \ge 0$ if the path is not available ($Z_k^s = 0$). With constraints (28), $V_k^s = 0$.
3. $V_k^s \ge 0$ if the path is available ($Z_k^s = 1$) and for any available path $p_{k'}^s$ that is available ($Z_{k'}^s = 1$) and $\phi_{kk'}^s = 0$. With constraints (26), $V_k^s = 0$.

From Proposition 1 and the definition of Φ^s, we have $V_k^s = 1$ if $\phi_{kk'}^s = 1$ for all the available paths that $Z_{k'}^s = 1$ and $V_k^s = 0$ if any available path that $Z_{k'}^s = 1$ has $\phi_{kk'}^s = 0$. This is equivalent to the above interpretation of the constraints, which completes the proof. $\square$

Then we can formulate the robust network design problem with box error $\mathcal{E}_B$ as follows:

$$\min_{X,y,Z,V} \sum_{s\in\mathcal{S}} \sum_{k\in\mathcal{K}} n^s \rho_k^s X_k^s + \alpha \sum_{(i,j)\in\mathcal{A}} (1 - y_{ij}), \tag{29}$$

$$\text{s.t.} \sum_{k\in\mathcal{K}} X_k^s = 1, \quad \forall s \in \mathcal{S}, \tag{30}$$

$$X_k^s \le V_k^s, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, \tag{31}$$

$$X_k^s \ge V_k^s - \sum_{n=1}^{k-1} V_n^s, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, \tag{32}$$

$$Z_{k'}^s \le y_{ij}, \quad \forall s \in \mathcal{S}, k' \in \mathcal{K}, (i,j) \in \mathcal{A}_k^s, \tag{33}$$

$$Z_{k'}^s \ge \sum_{(i,j)\in\mathcal{A}_k^s} y_{ij} - N_{k'}^s + 1, \quad \forall s \in \mathcal{S}, k' \in \mathcal{K}, \tag{34}$$

$$V_k^s \le \phi_{kk'}^s + (1 - Z_{k'}^s), \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, k' \in \mathcal{K}, \tag{35}$$

$$V_k^s \ge \sum_{k'\in\mathcal{K}} [(\phi_{kk'}^s - 1) Z_{k'}^s] + Z_k^s, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, \tag{36}$$

$$V_k^s \le Z_k^s. \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, \tag{37}$$

$$X_k^s, V_k^s \in \{0,1\}, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, \tag{38}$$

$$Z_{k'}^s \in \{0,1\}, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, \tag{39}$$

$$y_{ij} \in \{0,1\}, \quad \forall (i,j) \in \mathcal{A}. \tag{40}$$

The objective (29) of the model is to minimize the worst objective value desired by the leader among the paths that can be regarded as shortest under perception error $\mathcal{E}_B$ while minimizing the number of closed links. Constraints (30) guarantee only one path is chosen for each OD pair. Constraints (31) allow carriers to choose paths that can be perceived as shortest while constraints (32) restrict carriers to choose the worst case (smallest index) path. Constraints (33) forbid a carrier from taking the path if any of its arcs are closed while constraints (34) allow a path to be available if all of its arcs are open. Constraints (35)–(37) are from Proposition 2. Constraints (35) rule out path $p_k^s \in \mathcal{P}^s$ if $L_k^s > L_{k'}^s$ for any available paths in $\mathcal{P}^s$ while constraints (36) ensure path $p_k^s \in \mathcal{P}^s$ is perceived as the shortest path if $L_k^s \le L_{k'}^s$ for all available paths in $\mathcal{P}^s$. Constraints (37) require choosing from the available paths only. Constraints (38)–(40) ensure the decision variables are binary.

Path-Based Formulation with Both $\mathcal{E}_H$ and $\mathcal{E}_B$

After providing the formulations of path-based models with $\mathcal{E}_H$, $\mathcal{E}_B$ separately, we now provide a path-based formulation considering $\mathcal{E}_H$ and $\mathcal{E}_B$ together with minor modification of the path-based formulation with $\mathcal{E}_B$. The formulation is as the following:

$$\min_{X,y,Z,V} \sum_{s\in\mathcal{S}} \sum_{k\in\mathcal{K}} n^s \rho_k^s X_k^s + \alpha \sum_{(i,j)\in\mathcal{A}} (1 - y_{ij}),$$

$$(30) - (40),$$

$$X_k^s \sum_{(i,j)\in\mathcal{A}_k^s} c_{ij}^s \le V_{k'}^s (1+\kappa) \sum_{(i,j)\in\mathcal{A}_{k'}^s} c_{ij}^s, \quad \forall s \in \mathcal{S}, k \in \mathcal{K}, k' \in \mathcal{K}, \tag{41}$$

$$X_k^s \sum_{(i,j)\in\mathcal{A}_k^s} c_{ij}^s \le V_{k'}^s\Bigg(\sum_{(i,j)\in\mathcal{A}_{k'}^s} c_{ij}^s + E\Bigg), \quad \forall s\in\mathcal{S}, k\in\mathcal{K}, k'\in\mathcal{K}. \tag{42}$$

This formulation has two set of additional constraints (41) and (42) compared to the path-based formulation with $\mathcal{E}_B$. Constraints (41) enforce the chosen path is lower than $1+\kappa$ of the path cost for all paths in the generalized bounded rationality set with $\mathcal{E}_B$. Constraints (42) restrict the chosen path is lower than a constant E pluses the path cost for all paths in the generalized bounded rationality set with $\mathcal{E}_B$. These additional sets of constraints enforce the chosen path length to satisfy the hybrid bounded rationality set definition.

Path-Based Formulation with $\mathcal{E}_E$

The ellipsoidal set $\mathcal{E}_E$ is somewhat similar to $\mathcal{E}_H$, since it does not constrain each element ε_{ij}, but the whole vector ε collectively as in

$$||Q^{-1/2}\varepsilon||_2 \le \xi. \tag{43}$$

When path $\bar{p}$ with flow vector $\bar{x}$ is a solution to the perception-error model (4) for some $\varepsilon\in\mathcal{E}_E$, Sun et al. [17] show that

$$\sum_{(i,j)\in\mathcal{A}} c_{ij}\bar{x}_{ij} \le c^0 + \xi\sqrt{\bar{x}^T Q \bar{x}}. \tag{44}$$

We can use (44) instead of $\min\{c^0+E, (1+\kappa)c^0\}$ as in the case of $\mathcal{E}_H$ to generate sets Ψ_k^s. Not all paths whose lengths satisfy (44), however, are eligible to be included in Ψ_k^s. Some of them may not be perceived as the shortest path under the PE model with $\varepsilon\in\mathcal{E}_E$.

For each $\varepsilon\in\mathcal{E}_E$, a given path vector $\bar{x}\in\mathbb{X}$ to be an optimal solution to PE, there should exist a dual vector π such that

$$\bar{x}_{ij}(c_{ij}-\varepsilon_{ij}+\pi_i-\pi_j) = 0 \qquad \forall (i,j)\in\mathcal{A} \tag{45}$$

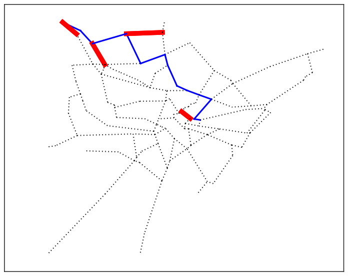
$$c_{ij}-\varepsilon_{ij}+\pi_i-\pi_j \ge 0 \qquad \forall (i,j)\in\mathcal{A} \tag{46}$$

Therefore (43), (45), and (46) must admit feasible ε and π for the given path vector $\bar{x}$ to be eligible to be included in the set Ψ_k^s. The feasibility check may be done by a convex optimization solver that can handle second-order constraints (43). Once the sets Ψ_k^s are constructed, the formulation remains the same as in the case of $\mathcal{E}_H$.

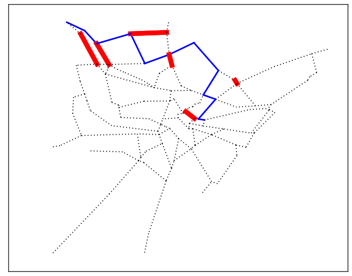
AN APPLICATION IN HAZARDOUS MATERIALS TRANSPORTATION

In this section, we show an application of the proposed concept of generalized bounded rationality in the hazmat network design problem. For the hazmat network design problem, the current literature assumes carriers always choose the minimum cost path. However, due to the unobservable attributes of the carriers and the lack of knowledge of how they make route decisions, this assumption is questionable. With the notation of generalized bounded rationality, we are able to assign route and link preferences for carriers, which could better capture how carriers make their route decisions.

For the illustration of the application, we use the set of data from the city of Ravenna, Italy [24]. The data consists of 105 nodes and 134 undirected arcs. Experiments are performed using C++ and CPLEX 12.6 on a computer with a Xeon processor and 32GB memory.

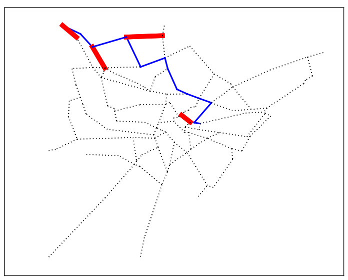


(a) $K = 200$, $R = 842.9$

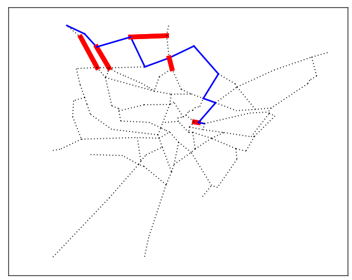


(b) $K = 500$, $R = 767.0$

FIGURE 2: Solution for the PHNDP-M-Sat Model. R is the worst-case risk measure.



(a) $K = 200$, $R = 842.9$



(b) $K = 500$, $R = 767.0$

FIGURE 3: Solution for the PHNDP-SM-Sat Model. R is the worst-case risk measure.

We run experiments on two cases of the path-based hazmat network design problem considering M-Sat (with PE set $\mathcal{E}_M$) driver behaviors (PHNDP-M-Sat) and SM-Sat (with PE set $\mathcal{E}_L$) driver behaviors (PHNDP-SM-Sat). For illustration, we first show the results on one OD pair. For both cases, we show the result when $K = 200$ and $K = 500$ as in Figure 2 and Figure 3. The value of R shows the risk value desired by the government (leader). When $K = 500$, the result is the same with that considering all the paths (obtained in Sun et al. [17]). As we can see, for $K = 500$, PHNDP-M-Sat closes one more link than the PHNDP-SM-Sat model since SM-Sat is more restrictive than M-Sat and the GBR path set with SM-Sat is a subset of GBR considering M-Sat. For the case of $K = 200$, we observe the risk value is 9.9% higher than that of the case $K = 500$. This is due to the limited path size that could be chosen by the carrier. If the government wants to limit the cost increase of the carrier, the path-based HNDP with GBR could achieve that by using a smaller K value. This is one advantage of the path-based models, which is the flexibility in choosing certain paths.

We also tested both the models by considering various numbers of OD pairs: 5, 10, 15 and 20. For each case, we show the results with a different number of paths: 50, 100, 150, 200 and 250. The result is shown in Table 1. The value under the "OD Num" is the risk value considering all paths obtained in Sun et al. [17]. For each model, we record the objective risk value, the risk gap compared to the risk value considering all paths, and the time to solve the model in seconds. We set a time limit of 2 hours. As can be seen, when the number of

TABLE 1: Solutions Characteristics for Path-Based Reformulations

OD Num (Optimal Risk)	# of Paths	PHNDP-M-Sat			PHNDP-SM-Sat		
		Risk	RiskGap%	Time (s)	Risk	RiskGap%	Time (s)
5 (253.0)	50	263.1	3.99	14.3	263.1	3.99	6.5
	100	263.1	3.99	37.0	263.1	3.99	15.2
	150	263.1	3.99	95.2	263.1	3.99	28.6
	200	263.1	3.99	193.7	263.1	3.99	38.6
	250	263.1	3.99	346.7	263.1	3.99	74.9
10 (434.6)	50	453.0	4.23	42.7	453.0	4.26	18.4
	100	439.4	1.1	185.4	439.4	1.13	57.0
	150	439.4	1.1	379.3	439.4	1.13	203.7
	200	439.4	1.1	915.3	439.4	1.13	522.3
	250	435.0	0.09	1351.8	435.0	0.12	429.0
15 (895.9)	50	919.5	2.63	85.0	916.8	2.75	29.6
	100	919.4	2.62	415.6	916.7	2.73	165.7
	150	919.4	2.62	876.5	902.7	1.17	615.9
	200	911.3	1.72	1834.6	909.9	1.97	579.4
	250	911.3	1.72	2241.2	898.5	0.69	3000.7
20 (999.6)	50	1075.2	7.56	135.8	1066.4	6.89	57.7
	100	1075.2	7.56	1050.5	1063.2	6.57	281.8
	150	1075.2	7.56	3338.1	1278.7	28.16	1084.2
	200	1081.8	8.22	> 7200	1075.0	7.75	2314.0
	250	1098.4	9.88	> 7200	1078.6	8.11	> 7200

OD pairs and paths gets larger, it grows much harder to solve the problem. In the cases we tested, three cases are solved sub-optimally.

By observing the results of model PHNDP-M-Sat, we see as the number of paths increases, the risk value is non-decreasing. This is intuitive as the more paths we consider, we can include paths with lower risk values. While we might include paths with higher risk as well, the model always restricts them from being chosen. For example, if the number of paths considered is 100, for the 100th path, model PHNDP-M-Sat would look beyond to find the multiplicative bounded rationality paths set for path 100 and restrict any path with higher risk from being chosen. We also notice that even with a small number of paths, the risk gap is relatively small. For the PHNDP-SM-Sat model, the risk trend in terms of increasing the number of paths is different. We see that some cases with a higher number of paths considered have higher risk gap. This is due to the strict restriction of the predefined number of paths. For example, if the number of paths considered is 100, for the 100th path, model PHNDP-SM-Sat assumes no paths beyond 100 would be considered. Thus with smaller number of paths, we could potentially omit higher risk paths, resulting in smaller risk value. In utilizing the PHNDP-SM-Sat, we should test cases whose number of paths is sufficiently large. Thus by comparison, even though the SM-Sat definition is more restrictive and potentially has a better interpretation, the PHNDP-M-Sat model is better in obtaining a more robust solution.

CONCLUSION

In this paper, we proposed path-based models for network design problems in which the drivers are boundedly rational. Particularly, we show path-based formulations with a hybrid perception error set $\mathcal{E}_H$, a box perception error set $\mathcal{E}_B$, the combination of $\mathcal{E}_H$ and $\mathcal{E}_B$, and an ellipsoidal perception error $\mathcal{E}_E$. The advantage of the path-based model is the structure of a single MIP formulation which could be solved by readily optimization solvers and the flexibility of defining which paths to consider, compared to the linked based model proposed by Sun et al. [17]. Further research in this area could be applying generalized bounded rationality with path-based models in the case of network design problems with the consideration of congestion.

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